Interior Point Methods for Semidefinite Programming

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for
Semidefinite Programming

PROEFSCHRIFT

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Contents

List of notation

$$
\begin{array}{rcl}
\mathbb{R}^n & : & n\text{-dimensional Euclidian vector space;} \\
\mathbb{R}^n_+ & : & \text{positive orthant of } \mathbb{R}^n; \\
\mathbb{R}^{n\times n} & : & \text{space of } (n\times n) \text{ real matrices;} \\
A^T & : & \text{transpose of } A\in\mathbb{R}^{m\times n}; \\
A_{ij} & : & ij\text{th entry of } A\in\mathbb{R}^{m\times n}; \\
\mathcal{S}_n & = & \left\{X \ : \ X\in\mathbb{R}^{n\times n},\ X=X^T\right\}; \\
A\succeq 0\ (A\succ 0) & : & A \text{ is symmetric positive semidefinite (positive definite);} \\
A\preceq 0\ (A\prec 0) & : & A \text{ is symmetric negative semidefinite (negative definite);} \\
\mathcal{S}_n^+ & = & \{X \ : \ X\in\mathcal{S}_n,\ X\succeq 0\}; \\
\text{int}\left(\mathcal{S}_n^+\right) & = & \{X \ : \ X\in\mathcal{S}_n,\ X\succ 0\}; \\
\lambda_i(A) & : & i\text{th largest eigenvalue of matrix } A\in\mathbb{R}^{n\times n}; \\
\lambda_{\max}(A) & = & \max\limits_i \lambda_i(A),\ \text{if } \lambda_i(A)\in\mathbb{R}\ \forall i; \\
\lambda_{\min}(A) & = & \min\limits_i \lambda_i(A),\ \text{if } \lambda_i(A)\in\mathbb{R}\ \forall i; \\
\mathbf{Tr}\,(A) & = & \sum\limits_i A_{ii}=\sum\limits_i \lambda_i(A) \text{ (trace of } A\in\mathbb{R}^{n\times n}\text{)}; \\
\det(A) & : & \text{determinant of } A\in\mathbb{R}^{n\times n}=\prod\limits_i \lambda_i(A); \\
\|A\|^2 & = & \mathbf{Tr}\,(AA^T)=\sum\limits_i\sum\limits_j A_{ij}^2 \text{ (Frobenius norm)} \\
 & = & \sum\limits_i \lambda_i^2(A) \text{ if } A\in\mathcal{S}_n; \\
\|A\|_2 & = & (\lambda_{\max}(A^TA))^{\frac{1}{2}} \text{ (spectral norm)}
\end{array}
$$

$$
\begin{array}{rcl}
 & = & \lambda_{\max}(A) \text{ if } A \succeq 0; \\
\rho(A) & = & \max_i |\lambda_i(A)| \text{ (spectral radius of } A\text{)}; \\
\kappa(A) & = & \dfrac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \text{ if } \lambda_i(A) > 0 \ \forall i \\
 & = & \text{condition number of } A \text{ if } A \succ 0; \\
\mathcal{R}(A) & : & \text{range (column space) of } A \in \mathbb{R}^{n \times n}; \\
A \sim B & : & \text{The matrices } A \text{ and } B \text{ are similar;} \\
\langle A, B \rangle & = & \mathbf{Tr} \left(AB^T \right); \\
A^{\frac{1}{2}} & = & \text{unique symmetric square root factor of } A \succeq 0; \\
\log(t) & = & \text{natural logarithm of } t; \\
\operatorname{diag}(x) & = & \text{diagonal matrix with components of } x \in \mathbb{R}^n \text{ on diagonal;} \\
\operatorname{diag}(X) & = & \text{vector obtained by extracting diagonal of } X \in \mathbb{R}^{n \times n}; \\
(P) & : & \text{primal problem in standard from;} \\
(D) & : & \text{Lagrangian dual problem of } (P); \\
\mathcal{P} & : & \text{feasible set of problem } (P); \\
\mathcal{D} & : & \text{feasible set of problem } (D); \\
\mathcal{P}^* & : & \text{optimal set of problem } (P); \\
\mathcal{D}^* & : & \text{optimal set of problem } (D); \\
(P_{gf}) & : & \text{ELSD dual of } (D); \\
(D_{cor}) & : & \text{Lagrangian dual of } (P_{gf}); \\
D & = & \left[X^{\frac{1}{2}} \left(X^{\frac{1}{2}} S X^{\frac{1}{2}} \right)^{-\frac{1}{2}} X^{\frac{1}{2}} \right]^{\frac{1}{2}} \text{ (Nesterov-Todd scaling matrix);} \\
\Psi(X, S) & = & -\log\det(XS) + n \log \mathbf{Tr}\,(XS) - n \log n \text{ (Centrality function);} \\
\phi(X, S) & = & (n + \nu\sqrt{n}) \log \mathbf{Tr}\,(XS) - \log\det(XS) - n \log n \\
 & \equiv & \text{Tanabe-Todd-Ye potential function;} \\
\mathbf{vec}(A) & = & [A_{11},\ A_{21}, \ldots A_{n1},\ A_{12},\ A_{22}, \ldots, A_{nn}]^T \text{ for } A \in \mathbb{R}^{n \times n}; \\
\mathbf{svec}(A) & = & [A_{11},\ \sqrt{2}A_{12},\ \ldots\ \sqrt{2}A_{1n},\ A_{22},\ \sqrt{2}A_{23},\ \ldots,\ A_{nn}]^T \text{ for } A \in \mathcal{S}_n;
\end{array}
$$

Chapter 1

Introduction

This thesis deals with algorithms for a subclass of nonlinear, convex optimization problems, namely semidefinite programs. In order place the topics which are dealt with in perspective, a short survey of the field of semidefinite programming[1] *is presented in this chapter. Introductory examples of applications in combinatorial optimization and engineering are described, after which interior point algorithms for this problem class are surveyed. A discussion on the scope of this thesis follows at the end of the chapter.*

1.1 Problem statement

One could easily be led to believe that the field of semidefinite programming (SDP) originated in this decade. A glance at a bibliography of SDP papers indeed indicates an explosion of research effort, starting around 1991. A closer look reveals that interest in this class of problems is somewhat older, and dates back to the 1960's (see *e.g.* [13]). A paper on SDP from 1981 is descriptively named *Linear Programming with Matrix Variables* [19], and this apt title may be the best way to introduce the problem.

[1] Some authors prefer the more descriptive term 'optimization' to the historically entrenched 'programming'.

The goal is to minimize the inner product

$$\langle C, X \rangle := \mathbf{Tr}\,(CX),$$

of two $n \times n$ symmetric matrices, a constant matrix C and a variable matrix X, subject to a set of constraints, where '**Tr**' denotes the trace (sum of diagonal elements) of a matrix.[2] The first of the constraints are linear:

$$\mathbf{Tr}\ (A_i X) = b_i, \quad i = 1, \ldots, m,$$

where the A_i's are given symmetric matrices, and the b_i's given scalars. Up to this point, the stated problem is merely a linear programming (LP) problem with the entries of X as variables. We now add the convex, nonlinear constraint that X must be symmetric positive semidefinite[3], denoted by $X \succeq 0$.[4]

The convexity follows from the convexity of the cone of positive semidefinite matrices.

The problem under consideration is therefore

$$(P): \quad p^* := \inf_X \left\{ \mathbf{Tr}\,(CX) \ : \mathbf{Tr}\,(A_i X) = b_i \ (i = 1, \ldots, m),\ X \succeq 0 \right\},$$

which has an associated Lagrangian dual problem:

$$(D): \quad d^* := \sup_{y,S} \left\{ b^T y \ : \ \sum_{i=1}^m y_i A_i + S = C,\ S \succeq 0,\ y \in \mathbb{R}^m \right\}.$$

The duality theory of SDP is weaker than that of LP. One still has the familiar *weak duality* property: Feasible X, y, S satisfy

$$\mathbf{Tr}\,(CX) - b^T y = \mathbf{Tr}\ \left(\left(S + \sum_{i=1}^m y_i A_i \right) X \right) - \sum_{i=1}^m y_i \mathbf{Tr}\,(A_i X) = \mathbf{Tr}\,(SX) \geq 0,$$

[2]This inner product corresponds to the familiar Euclidean inner product of two vectors – if the columns of the two matrices C and X are stacked to form vectors **vec**(X) and **vec**(C), then **vec**$(C)^T$**vec**$(X) =$ **Tr** (CX). The inner product induces the so-called *Frobenius norm*:

$$\|A\|^2 := \langle A, A \rangle = \mathbf{Tr}\ \left(AA^T\right) = \sum_{i,j} A_{ij}^2.$$

[3]By definition, a symmetric matrix X is positive semidefinite if $z^T X z \geq 0\ \forall z \in \mathbb{R}^n$, or equivalently, if all eigenvalues of X are nonnegative.

[4]The symbol '$\succeq$' denotes the so-called Löwner partial order on the symmetric matrices: $A \succeq B$ means $A - B$ is positive semidefinite.

where the inequality follows from $X \succeq 0$ and $S \succeq 0$. In other words, the *duality gap* is nonnegative for feasible solutions.

Solutions (X, y, S) with zero duality gap

$$\mathbf{Tr}\,(CX) - b^T y = \mathbf{Tr}\,(SX) = 0$$

are optimal. For LP, if either the primal or the dual problem has an optimal solution, then both have optimal solutions, and the duality gap at optimality is zero. This is the *strong duality* property. The SDP case is more subtle: One problem may be solvable and its dual infeasible, or the duality gap may be positive at optimality, etc. The existence of primal and dual optimal solutions is only guaranteed if both (P) and (D) allow positive definite solutions, *i.e.* feasible $X \succ 0$ and $S \succ 0$. This is called the *Slater constraint qualification* (or Slater regularity condition). These duality issues will be discussed in detail in Chapter 2.

1.2 The importance of semidefinite programming

SDP problems are of interest for a number of reasons, including:

- SDP contains important classes of problems as special cases, such as linear and quadratic programming (LP and QP);
- important applications exist in combinatorial optimization and mechanical and electrical engineering;
- efficient, polynomial time solution strategies (interior point methods) have emerged in the past few years (explaining the resurgence in research interest).

Each of these considerations will be discussed briefly in the remainder of this chapter. The presentation here is based on the short survey by De Klerk *et al.* [23], and is designed to minimize the overlap with other surveys on SDP: The seminal work of Nesterov and Nemirovskii [84] contains a section on special cases of SDP problems (§6.4), as well as the development of an entire interior point methodology. An excellent survey by Vandenberghe and Boyd [112] deals with basic theory, diverse applications, and so-called potential reduction algorithms (up to 1995). Two more recent surveys which focus more on applications of SDP in

combinatorial optimization are by Alizadeh [2] and by Ramana and Pardalos [95]. The former also deals with interior point methodology, whilst the latter contains surveys of geometric properties of the SDP feasible set (so-called spectrahedra), as well as complexity and duality theory. Lewis and Overton [66] give a nice historical perspective on the development of SDP and focus on the relation with eigenvalue optimization.

1.3 Special cases of semidefinite programming

If the matrix X is restricted to be diagonal, then the requirement $X \succeq 0$ reduces to the requirement that the diagonal elements of X must be nonnegative. In other words, we have an LP problem. Optimization problems with convex quadratic constraints are likewise special cases of SDP.[5] This follows from the well-known *Schur complement* reformulation: If

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

for an invertible A, then the matrix

$$S_{chur} := C - B^T A^{-1} B$$

is called the Schur complement of A in X. One has

$$\begin{aligned} X \succ 0 \quad &\text{if and only if} \quad A \succ 0 \text{ and } S_{chur} \succ 0, \\ \text{if } A \succ 0, \text{ then } X \succeq 0 \quad &\text{if and only if} \quad S_{chur} \succeq 0. \end{aligned}$$

It follows that we can represent the quadratic constraint

$$(Ax+b)^T(Ax+b) - (c^T x + d) \leq 0, \quad x \in \mathbb{R}^n,$$

by the semidefinite constraint

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^T x + d \end{bmatrix} \succeq 0.$$

[5] This includes the well-known convex quadratic programming (QP) problem.

In the same way, we can represent the *second order cone* (or 'ice cream cone'):

$$\left\{ (t,x) \ : \ t \geq \sqrt{\sum_{i=1}^{n} x_i^2} \right\},$$

by

$$\begin{bmatrix} tI & x \\ x^T & t \end{bmatrix} \succeq 0.$$

Another nonlinear example is

$$\min_x \left\{ \frac{(c^T x)^2}{d^T x} \ : \ Ax \geq b \right\},$$

where it is known that $d^T x > 0$ if $Ax \geq b$. An equivalent SDP problem is:[6]

$$\min_{t,x} \left\{ t \ : \ \begin{bmatrix} t & c^T x & 0 \\ c^T x & d^T x & 0 \\ 0 & 0 & \operatorname{diag}(Ax-b) \end{bmatrix} \succeq 0 \right\}.$$

Several problems involving matrix norm or eigenvalue minimization may be stated as SDP's. A list of such problems may be found in [112]. A simple example is the classical problem of finding the largest eigenvalue $\lambda_{\max}(A)$ of a symmetric matrix A. The key observation here is that $t \geq \lambda_{\max}(A)$ if and only if $tI - A \succeq 0$. The SDP problem therefore becomes

$$\min_t \{ t \ : \ tI - A \succeq 0, \quad t \in \mathbb{R} \}.$$

An SDP algorithm for this problem is described in [46, 48].

1.4 Applications in combinatorial optimization

General quadratic optimization problems allow SDP relaxations. The key observation is that

$$x^T Q x = \mathbf{Tr}\,(Q x x^T),$$

[6] We use the notation 'diag' as follows: for a matrix X, $\operatorname{diag}(X)$ is the vector obtained by extracting the diagonal of X; for a vector x, $\operatorname{diag}(x)$ is the diagonal matrix with the coordinates of x as diagonal elements.

for a given matrix Q and vector x. The rank one matrix $X := xx^T$ is positive semidefinite. We can therefore relax the condition $X = xx^T$ to $X \succeq 0$. This relaxation is originally due to Shor [102].

Combinatorial optimization problems can in turn be written as quadratic optimization problems. The condition $x_i \in \{-1, 1\}$ is equivalent to $x_i^2 = 1$, for example. It is not immediately obvious that one can benefit from this non-convex problem reformulation. However, the Shor relaxation of quadratically reformulated combinatorial problems has become a powerful theoretical and computational tool. Lovász and Schrijver [68] considered the generic combinatorial problem

$$q^{\max} = \max \left\{ x^T Q x \ : \ x_i \in \{-1, 1\} \ (\forall i) \right\}, \tag{1.1}$$

and suggested the relaxation[7]

$$\bar{q} = \max \left\{ \mathbf{Tr}\,(QX) \ : \ \mathrm{diag}\,(X) = e, \ X \succeq 0 \right\}. \tag{1.2}$$

For this general relaxation Nesterov [83] recently proved that

$$\bar{q} - \underline{q} \geq q^{\max} - q^{\min} \geq \frac{4 - \pi}{\pi} \left(\bar{q} - \underline{q} \right)$$

where $(q^{\min}, q^{\max})$ is the range of feasible objective values in (1.1), and $(\underline{q}, \bar{q})$ is the range of feasible values in the relaxation problem (1.2). Moreover, a random feasible solution x to (1.1) can be computed from the solution to the relaxation. The expected objective value of x, say $E(x)$, satisfies[8]

$$\frac{q^{\max} - E(x)}{q^{\max} - q^{\min}} < \frac{4}{7}.$$

For specific problems this bound can be improved. The showcase example is the *maximal cut problem, i.e.* the problem of finding a cut of maximal weight through a graph with weighted edges. In a pioneering article, Goemans and Williamson [36] proved that $\bar{q} \leq 1.14 q^{\max}$ in this case. They moreover devised a randomized algorithm which produces a cut with expected value greater than $0.878 q^{\max}$. Practical experience with the method is also positive — the solutions (cuts) obtained in

[7] Note that if $x_i \in \{-1, 1\}$ then $\mathrm{diag}\,(xx^T) = e$.

[8] The same bounds were obtained by Ye [118] for the 'box-constrained' problem where $x_i \in \{-1, 1\}$ is replaced by $-1 \leq x_i \leq 1$ in problem (1.1); these results were further extended in [119] to include *simple quadratic constraints* of the form: $\sum_{i=1}^n a_i x_i^2 = b$.

this way are often much closer to optimality than predicted (even optimal), and the optimal solution is easily found in conjunction with a branch and bound scheme, if necessary [96].

Similar bounds were also proved in [36] for satisfiability problems.

The SDP relaxations are not always useful, though. Cases where the SDP relaxation is no stronger than the usual LP relaxation are reviewed in [95].

SDP offers more than just a numerical tool to generate lower and upper bounds on optimal values. It also provides a technique of proof via duality theory. We consider the classical *sandwich theorem*, and include a simple proof (which is new to the best of our knowledge) which nicely illustrates some basic features of SDP: how the Shor relaxations can be devised, how strong duality may be utilized, and how SDP problems are often related to eigenvalue optimization.

The sandwich theorem relates three characterizing numbers of a graph $G(V, E)$: the colouring number[9] $\chi(G)$, the maximal clique number[10] $\omega(G)$, and the Lovász number $\theta(G)$, which can be defined[11] as the optimal value of the following SDP relaxation of the maximal clique problem (see [67, 42]):

$$\theta(G) := \max \mathbf{Tr}\left(ee^T X\right) = e^T X e \tag{1.3}$$

subject to

$$\left.\begin{array}{rcl} X_{ij} & = & 0,\ \{i,j\} \notin E\ (i \neq j) \\ \mathbf{Tr}\,(X) & = & 1 \\ X & \succeq & 0. \end{array}\right\} \tag{1.4}$$

The sandwich theorem states the following.

Theorem 1.4.1 (Lovász's Sandwich Theorem) *For any graph $G = (V, E)$ one has*

$$\omega(G) \leq \theta(G) \leq \chi(G).$$

[9] Number of colours needed to colour all vertices so that no two adjacent vertices share the same colour.

[10] A maximal clique (completely connected subgraph) is a subset $C \subset V$ with $\forall i, j \in C(i \neq j) \ : \{i, j\} \in E$, such that $|C|$ is maximal. The cardinality $|C|$ is called the maximal clique number.

[11] Strictly speaking, the definition given here is of the Lovász number of the complement of G (nodes in the complement of G are connected if and only if they are not connected in G).

Proof:

In order to prove the first inequality of the theorem, we must show that problem (1.3) is indeed a relaxation of the maximal clique problem.

Let x_C denote a 0-1 vector which defines a clique C of size k in G, i.e:

$$(x_C)_i = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the rank one matrix

$$X := \frac{1}{k} x_C x_C^T$$

is feasible in (1.4) with objective value

$$e^T X e = \frac{1}{k} \left(e^T x_C \right)^2 = \frac{k^2}{k} = k.$$

We therefore have $\omega(G) \leq \theta(G)$, which is the first part of the sandwich theorem.

The second part is to prove $\theta(G) \leq \chi(G)$. To this end, we write down the Lagrangian dual of the SDP relaxation (1.4) to obtain

$$\theta(G) = \min \lambda \tag{1.5}$$

subject to

$$\left.\begin{array}{rcl} Y + ee^T & \preceq & \lambda I \\ Y_{ij} & = & 0,\ \{i,j\} \in E\ (i \neq j) \\ Y_{ii} & = & 0,\ i \in V. \end{array}\right\} \tag{1.6}$$

Note that both the primal problem (1.3) and dual problem (1.5) satisfy the Slater constraint qualification, and therefore have the same optimal value, namely $\theta(G)$.

Given a colouring of G with k colours, we must construct a feasible solution for (1.6) with $\lambda \leq k$. Such a colouring defines a partition $V = \cup_{i=1}^{k} C_i$ where the C_i's are subsets of nodes sharing the same colour. In other words, the C_i's must be disjoint stable sets (co-cliques). Now let $\gamma_i = |C_i|$ and define

$$M_i := k \left(I_{\gamma_i} - J_{\gamma_i} \right), \quad i = 1, \ldots, k,$$

where I_{γ_i} is the $(\gamma_i \times \gamma_i)$ identity matrix, and J_{γ_i} the all-one matrix of the same size.

We will show that the block diagonal matrix

$$Y = \begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & M_k \end{pmatrix} \tag{1.7}$$

is feasible in (1.6) if $\lambda = k$. By construction, Y satisfies the last two constraints in (1.6). We must still show that $Y + ee^T \preceq kI$, *i.e.* the largest eigenvalue of $Y + ee^T$ must be at most k. This proof is not difficult but requires some linear algebra, and is given in Appendix A (Lemma A.2.1). This completes the proof of the sandwich theorem. □

Moreover, we have given a proof of the equivalence of two different definitions of $\theta(G)$ via (1.3) and (1.5).[12] It is interesting to note that the definition of $\theta(G)$ as an SDP problem dates back to 1979. As such it is a fine example of a historical problem which has benefited from the emergence of efficient solution algorithms in the 1990's. More examples follow in the next section.

1.5 Engineering applications

The richest field of application of SDP is currently *system and control theory*, where SDP has become a established tool. The standard reference for these problems is Boyd *et al.* [17]. Introductory examples are given in [112] and [86]. The latter reference deals with a problem in *active noise control*: The noise level inside a dome is reduced by emitting sound waves at the same frequency but with a suitable phase shift. The underlying control problem involves optimization over the second order cone, which was shown to be a special case of SDP in Section 1.3.

[12] These and other equivalent definitions of $\theta(G)$ are discussed in [42].

Other engineering applications of SDP include: *VLSI transistor sizing*, *pattern recognition* using ellipsoids, and *logarithmic Chebychev approximation* (see [112]). The usefulness of Shor relaxations are also being investigated for nonlinear combinatorial optimization problems involving *optimal nuclear reactor reloading* (see De Klerk *et al.* [20]).

An application which receives less attention is *structural design*, where the best known SDP problem involves optimal truss[13] design. Two variants are:

1. minimize the weight of the structure such that its fundamental frequency[14] remains above a critical value;

2. minimize the worst-case compliance ('stored energy') of the truss given a set of forces which the structure has to withstand.

The second of these problems allows another nice application of SDP duality theory. The problem may be stated as (see *e.g.* [16]):

Displacement formulation

$$\min_{t,x_1,\ldots,x_k} \left\{ \max_{j=1,\ldots,k} \left\{ x_j^T f_j \right\} \right\}$$

subject to

$$\begin{aligned} \left(\sum_{i=1}^{m} t_i \frac{E_i}{l_i^2} b_i b_i^T \right) x_j &= f_j, \quad j = 1, \ldots, k \\ \sum_{i=1}^{m} t_i &= V, \quad t \geq 0, \end{aligned}$$

where the t_i's are the bar volumes (design variables), and the f_j's are the set of forces which the truss has to withstand. The displacement of the nodes subject to force f_j is given by the vector x_j. The fixed vectors b_i depend only on the layout of the nodes, and E_i and l_i denote the Young modulus and length of bar i, respectively.

[13] A truss is here defined as a structure of bars which connect a fixed ground structure of nodes. (A famous example is the Eiffel tower!) The design is fixed once the sizes of the bars have been decided.

[14] Frequency at which the structure resonates.

The first constraint requires equilibrium of the structure and the second fixes its total volume. The objective is to minimize the worst-case compliance. Loosely speaking, one would like to maximize the 'stiffness' of the structure. The matrix

$$K(t) := \left[\sum_{i=1}^{m} t_i \frac{E_i}{l_i^2} b_i b_i^T\right]$$

is also called the *stiffness matrix* of the structure — it gives the relation between the displacements and the applied force via $K(t)x_j = f_j$.

The name 'displacement' formulation stems from the displacement variables x_j. From engineering considerations, the problem may also be stated by using the forces in the bars as variables (see *e.g.* [16]).

Bar forces formulation

$$\min_{\beta,t} \left\{ \max_{j=1,\ldots,k} \left\{ \sum_{\substack{i=1 \\ t_i>0}}^{m} \frac{1}{2} \frac{l_i^2}{E_i} \frac{\beta_{ij}^2}{t_i} \right\} \right\}$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} t_i &= V \\ f_j &= \sum_{i=1}^{m} \beta_{ij} b_i, \quad j = 1, \ldots, k \\ t_i &\geq 0, \quad i = 1, \ldots, m \\ \beta_{ij} = 0 \quad \text{if } t_i &= 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, k, \end{aligned}$$

where β_{ij} is the reaction force in bar i due to the external force f_j. The second constraint simply requires static equilibrium, *i.e.* the reaction forces in the bars must balance the external loads. The objective function describes the so-called *complementary energy* of the bars. Here the stiffness of the truss is maximized by minimizing the worst-case complementary energy.

From a purely mathematical point of view it is far from obvious that the two formulations are equivalent. This equivalence can be shown using SDP duality. We will sketch the proof here. Using the Schur complement trick, the displacement formulation can be written as an SDP problem (for details, see Nemirovskii [80] or De Klerk *et al.* [21]).

SDP reformulation of the displacement formulation

$$\min \tau$$

subject to

$$\begin{aligned} \begin{bmatrix} \tau & f_j^T \\ f_j & \left(\sum_{i=1}^m t_i \frac{E_i}{l_i^2} b_i b_i^T\right) \end{bmatrix} &\succeq 0, \quad j = 1, \ldots, k \\ \sum_{i=1}^m t_i &= V, \quad t \geq 0. \end{aligned}$$

The equivalence proof is now done in three steps:

1: write down the dual of the SDP reformulation and simplify it;

2: obtain the dual of the resulting problem from Step 1;

3: reduce the problem obtained in Step 2 to the 'bar forces' formulation.

This sequence of steps is described in detail in [80]. A survey of these and related formulations is given by De Klerk *et al.* [21], with emphasis on SDP formulations. The computational advantages of using interior point methods to solve these problem reformulations have been demonstrated by Ben-Tal and Nemirovskii [14], by solving the largest problem instances to date.

An important observation is that the bar forces formulation can be reduced to an LP problem if only one external force is considered [16]. Thus large problem instances have been solved in the past using interior point (and Simplex based) methods for LP. The development of interior point methods for semidefinite programming has now extended the range of tractable problems to include multiple load scenarios.

Other structural design problems which may be formulated as SDP's include sandwich plate design [15], optimization of variable thickness sheets [100], and minimal compliance design with optimized materials [97]. Good reviews of interior point methods in truss topology design are Bendsøe *et al.* [16], and Jarre *et al.* [55].

1.6 Interior point methods

Bearing the links between LP and SDP in mind, it may come as little surprise that interior point algorithms for LP have been successfully extended to SDP.

The field of interior point methods for LP more or less started with the *ellipsoid algorithm* of Khacijan [59] in 1979, which allowed a polynomial bound on the worst-case iteration count. This resolved the question whether linear programming problems are solvable in so-called polynomial time, but practical experiences with the ellipsoid method were disappointing. The next major development was the famous paper by Karmarkar [58] in 1984, which introduced an algorithm with an improved complexity bound which was also accompanied by claims of computational efficiency. In the following decade several thousand papers appeared on this topic. A major survey of interior point methods for LP was done by Gonzaga [40] (up to 1992); some more recent review papers include [34] and [48]. Several new books on the subject have also appeared recently, including Roos *et al.* [98] and Wright [114]. It has taken nearly ten years to substantiate the claims of the computational efficiency of interior point methods; several studies have now indicated that these methods have superiour performance to state-of-the-art Simplex algorithms on large scale problems (see *e.g.* Lustig *et al.* [71] and most recently Andersen and Andersen [6]).

The first extension of interior point algorithms from LP to SDP was by Nesterov and Nemirovskii [84], and independently by Alizadeh [1] in 1991. Nesterov and Nemirovskii actually considered convex optimization problems in the generic *conic formulation*:

$$\min_x \left\{ c^T x \ : \ x \in (L + b) \cap C \right\}, \tag{1.8}$$

where L denotes a linear subspace of $\mathbb{R}^n$, $b, c \in \mathbb{R}^n$, and C is a closed and pointed[15] convex cone with nonempty interior. The associated dual problem is

$$\max_y \left\{ b^T y \ : \ y \in (L^\perp + c) \cap C^* \right\},$$

where $L^\perp$ is the orthogonal complement of L in $\mathbb{R}^n$, and C^* is the *dual cone* of C:

$$C^* := \{ x \mid \langle x, y \rangle \geq 0 \ \forall \ y \in C \}.$$

Note that the nonlinearity in the problem is 'banished' to a convex cone.[16] In the SDP case this cone is the cone of semidefinite matrices:

$$\mathcal{S}_n^+ := \{ X \ : \ X \in \mathcal{S}_n, \ X \succeq 0 \},$$

[15] A cone is called pointed if it contains no lines.

[16] All convex optimization problems can in principle be cast in the conic form [84].

where $\mathcal{S}_n$ denotes the space of symmetric $n \times n$ matrices. Nesterov and Nemirovskii showed that such conic optimization problems can be solved by sequential minimization techniques, where the conic constraint is discarded and a barrier term is added to the objective. Suitable barriers are called *self-concordant*. These are smooth convex functions with second derivatives which are Lipschitz continuous with respect to a local metric (the metric induced by the Hessian of the function itself). Self-concordant barriers go to infinity as the boundary of the cone is approached, and can be minimized efficiently by Newton's method.[17] Each convex cone C possesses a self-concordant barrier, although such barriers are only computable for some special cones. The function $f(x) = -\sum_{i=1}^{n} \log(x_i)$ is such a barrier for the positive orthant of $\mathbb{R}^n$, and is instrumental in designing interior point methods for LP. Likewise, the function

$$f_{bar}(X) = -\log\det(X)$$

is a self-concordant barrier for the cone of semidefinite matrices [84]. Using this barrier, several classes of algorithms may be formulated which have polynomial worst-case iteration bounds for the computation of ϵ-optimal solutions, *i.e.* feasible (X^*, S^*) with duality gap $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$, where $\epsilon > 0$ is a given tolerance.

1.6.1 Logarithmic barrier methods

Primal log-barrier methods use Newton's method to solve a sequence of problems of the form

$$\min_X \left\{ \mathbf{Tr}\,(CX) - \mu \log\det(X) \;:\; \mathbf{Tr}\,(A_i X) = b_i \;(i = 1, \ldots, m) \right\},$$

where the parameter μ is sequentially decreased to zero. Such algorithms were analysed by Faybusovich in [30, 31] and later by other authors in [43] and [10]. Note that the condition $X \succeq 0$ has been replaced by adding a 'barrier term' to the objective.[18] The condition $X \succeq 0$ is maintained by controlling the Newton process carefully — large decreases of μ necessitate damped Newton steps (see *e.g.* [10]), while small updates allow full Newton steps (see *e.g.* [43]).

[17] The definition of self-concordant barriers will not be used here and is omitted; a well-written introductory text dealing with self-concordance is [54].

[18] This idea actually dates back to the 1960's and the work of Fiacco and McCormick [32]; the implications for complexity theory only became clear two decades later, when Gill *et al.* [35] showed that the method of Karmarkar could be interpreted as a logarithmic barrier method.

Following the trend in LP, so-called *primal-dual* methods soon became more popular. These methods minimize the duality gap

$$\mathbf{Tr}\,(CX) - b^T y = \mathbf{Tr}\,(XS),$$

and employ the combined primal-dual barrier function

$$f_{pd} := -\left(\log\det(X) + \log\det(S)\right) = -\log\det(XS).$$

This means that a sequence of problems of the following form are solved

$$\min_{X,y,S}\left\{\mathbf{Tr}\,(XS) - \mu\log\det(XS) \;:\; \mathbf{Tr}\,(A_i X) = b_i\ (i=1,\ldots,m),\ \sum_{i=1}^{m} y_i A_i + S = C\right\}. \tag{1.9}$$

The minimizers for (1.9) satisfy

$$\left.\begin{array}{rcll} \mathbf{Tr}\,(A_i X) & = & b_i, & i=1,\ldots,m \\ \sum_{i=1}^m y_i A_i + S & = & C & \\ XS & = & \mu I & \\ X, S & \succ & 0. & \end{array}\right\} \tag{1.10}$$

These equations can be viewed as a perturbation of the optimality conditions of (P) and (D), where $\mu = 0$. System (1.10) has a unique solution under the assumptions that the A_i's $(i = 1,\ldots,m)$ are linearly independent, and that the feasible sets of (P) and (D) have nonempty interiors. This solution will be denoted by $X(\mu), S(\mu), y(\mu)$, and may be seen interpreted as the parametric representation of a smooth curve (the *central path*) in terms of the parameter μ. The central path converges to the so-called *analytic center* of the optimal set as $\mu \to 0$ (see Chapter 2). This result was first obtained by Goldfarb and Scheinberg [37], although it was already shown by De Klerk *et al.* [22] that all limit points of sequences along the central path lie in the relative interior of the optimal set.

Logarithmic-barrier methods are also called *path-following* methods, due to the relation between the central path and the log-barrier function.

Primal-dual log-barrier methods solve the system (1.10) approximately, followed by a reduction in μ. Ideally, the goal is to obtain primal and dual steps ΔX and

ΔS, respectively, which satisfy $X + \Delta X \succeq 0$, $S + \Delta S \succeq 0$ and

$$\begin{aligned} \mathbf{Tr}\,(A_i \Delta X) &= 0, \quad i = 1, \ldots, m \\ \textstyle\sum_{i=1}^{m} \Delta y_i A_i + \Delta S &= 0 \\ (X + \Delta X)(S + \Delta S) &= \mu I. \end{aligned} \tag{1.11}$$

The last equation is nonlinear, and primal-dual methods differ with regard to how it is linearized. Moreover, care must be taken to ensure that the solution matrices ΔX and ΔS are symmetrical. Zhang [121] suggested to replace the nonlinear equation by

$$H_P\left(\Delta X S + X \Delta S\right) = \mu I - H_P(XS), \tag{1.12}$$

where H_P is the linear transformation given by

$$H_P(M) := \frac{1}{2}\left[PMP^{-1} + P^{-T} M^T P^T\right],$$

for any symmetric matrix M, and where the *scaling matrix* P determines the symmetrization strategy. Some popular choices for P are listed in Table 1.1. The

P	Reference
$\left[X^{\frac{1}{2}}\left(X^{\frac{1}{2}} S X^{\frac{1}{2}}\right)^{-\frac{1}{2}} X^{\frac{1}{2}}\right]^{\frac{1}{2}}$	Nesterov and Todd [85];
$X^{-\frac{1}{2}}$	Monteiro [73], Kojima *et al.* [65];
$S^{\frac{1}{2}}$	Monteiro [73], Helmberg *et al.* [44], Kojima *et al.* [65];
I	Alizadeh, Haeberley and Overton [4];

Table 1.1: Choices for the scaling matrix P.

proof of the existence and uniqueness of each of the resulting search directions was done by Shidah *et al.* in [101].[19] Other properties (such as scale-invariance) are compared by Todd *et al.* in [108].

[19] For $P = I$ uniqueness is not always guaranteed; a sufficient condition for uniqueness is $XS + SX \succeq 0$.

The conspicious entry $P = \left[X^{\frac{1}{2}}\left(X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}}$ in Table 1.1 warrants some comment. Nesterov and Todd [85] showed[20] that for each pair $X \succ 0$, $S \succ 0$ there exists a matrix D such that

$$\nabla^2 f_{bar}(D)X = S.$$

It is shown in Appendix A that the Hessian $\nabla^2 f_{bar}(D)$ is the linear operator which satisfies

$$\nabla^2 f_{bar}(D) : X \mapsto D^{-1}XD^{-1}.$$

It follows that $X = DSD$, from which it easily follows that $D = P^2$. In this way we obtain the *symmetric primal-dual scaling* $P^{-1}XP^{-1} = PSP$. This symmetry explains the usefulness of D in symmetrization.

Algorithms differ in how μ is updated, and how the symmetrized equations are solved. Methods with use large reductions of μ followed by several damped Newton steps are called long step (or large update) methods. These are analysed in [57], [73], and [104].

Methods which use dynamic updates of μ include the popular *predictor-corrector* methods. References include [4, 61, 64, 90, 106]. The dynamic updates in De Klerk *et al.* [26] utilize an improved analysis of a new measure of centrality (distance to the central path).

Superlinear convergence properties of predictor-corrector schemes are studied in [61, 63, 70].

1.6.2 Affine–scaling methods

Affine–scaling algorithms for LP have been of interest since it became clear that Karmarkar's algorithm was closely related to the primal affine–scaling method of Dikin [29] (from 1967!). In fact, modifications of Karmarkar's algorithm by Vanderbei *et al.* [113] and Barnes [11] proved to be a rediscovery of the primal affine–scaling method.

The primal affine–scaling direction for SDP minimizes the primal objective over an ellipsoid which is inscribed in the primal feasible region. Surprisingly, Muramatsu [78] has shown that an SDP algorithm using this search direction may

[20] This result was actually proved in the more general setting of conic optimization problems where the cone C in (1.8) is self-dual, *i.e.* $C = C^*$. The interested reader is referred to [85].

converge to a non-optimal point, regardless of which step length is used. This is in sharp contrast to the LP case, and shows that extension of algorithms from LP to SDP cannot always be taken for granted.

Two primal-dual variants of the affine–scaling methods were extended by De Klerk *et al.* in [27] from LP to SDP. These algorithms minimize the duality gap over ellipsoids in the scaled primal-dual space, where the matrix $P = D^{\frac{1}{2}}$ is used for the scaling. One of the two methods is the *classical affine–scaling method*, where the search direction is obtained by using $\mu = 0$ in (1.12). The primal-dual affine–scaling method fails if the scalings $P = S^{\frac{1}{2}}$ or $P = I$ from Table 1.1 is used (instead of $P = D^{\frac{1}{2}}$). This was recently proved by Muramatsu and Vanderbei [79].

1.6.3 Primal-dual potential reduction methods

These algorithms are based on the potential function

$$\phi(X, S) = (n + \nu\sqrt{n}) \log (\mathbf{Tr}\,(XS)) - \log (\det(XS)) - n \log n,$$

where $\nu \geq 1$. In order to obtain a polynomial complexity bound it is sufficient to show that ϕ can be reduced by an absolute constant at each iteration [111]. A survey of algorithms which achieve such a reduction is given in [112].

1.6.4 Infeasible–start methods

Several infeasible-start SDP algorithms have been suggested. A review of traditional big-M initialization strategies is given in [112]. One of the first infeasible–start predictor-corrector algorithms was by Potra and Sheng [90]; this algorithm solves a *homogeneous embedding* of (P) and (D) using an infeasible-start algorithm. Recently the authors report some positive computational results in [91]. Other references for infeasible-start methods include [61, 70].

The idea of embedding the SDP problem in an extended self-dual problem with known starting point on the central path was investigated for SDP by De Klerk *et al.* [22] and independently by Luo *et al.* [69]. The idea of self-dual embeddings for LP dates back to the 1950's and the work of Goldman and Tucker [38]. With the arrival of interior point methods, the embedding idea was revived to be used in infeasible–start algorithms by Ye *et al.* [120].

A solution of the self-dual embedding gives information about the solution of the original problem. The SDP analysis was extended in De Klerk *et al.* [24] to include pathological cases caused by the weaker duality theory of SDP (as compared to LP). In the latter case the stronger ELSD (extended Lagrange-Slater) dual problem is used in the embedding. These duals have better properties than the usual Lagrangian duals, and were formulated by Ramana [92] (see also [94]).

1.7 The scope of this thesis

This thesis deals with each of the categories of interior point algorithms described in the previous section. The central idea is to provide a wide framework of algorithms for solving SDP problems without regularity assumptions (like Slater's constraint qualification), by employing the self–dual embedding strategy.

A comprehensive treatment of self–dual embeddings is given in Chapter 2, based on De Klerk *et al.* [22, 24]. The approach of this chapter offers a new perspective on how to systematically obtain as much information as possible about a given SDP problem, by solving successive embedding problems. The embedding approach relies on the fact that the central path converges to the so-called analytic center of the optimal set; a proof of this result (and related results) are therefore contained in the first part of the chapter. In the second part it is shown how to detect general infeasibility or unboundedness of an SDP problem, under certain assumptions; this requires the use of ELSD (instead of Lagrangian) duals in the embedding.

Algorithms to solve the embedding problem form the subject matter for the remainder of the thesis. Primal–dual affine–scaling methods are studied in Chapter 3, based on results by De Klerk *et al.* [27]. Two such methods are extended from LP to SDP: the classical primal–dual affine–scaling algorithm of Monteiro *et al.* [74], as well as the Dikin-type primal–dual affine–scaling algorithm of Jansen *et al.* [49]. The extended algorithms have the same worst-case iteration complexity bounds as their LP counterparts, and are the only globally convergent affine–scaling algorithms for SDP to date.

The methods of Chapter 3 are of significant theoretical importance, but fall in the class of *short step methods*, which are as such not suitable for implementation (the iteration count will always be close to the predicted worst-case behavior). Implementable variants of the algorithms are therefore studied in Chapter 4, based on

De Klerk *et al.* [28]. The resulting new methods fall in the category of potential reduction algorithms, and function as 'long step predictor-corrector' algorithms. A numerical comparison is presented with the potential reduction method of Nesterov and Todd [85], as implemented by Vandenberghe and Boyd [110].

In Chapters 5 and 6 the attention is shifted to logarithmic barrier (path following) methods. The primal method is analysed first, followed by the primal–dual variant in Chapter 6. The aim of Chapter 5 is twofold: the original analysis of Faybusovich [30, 31] is simplified, and primal methods are cast in a new light as solution algorithms of the embedding problem, for the case where the original problem is in the so-called *symmetric form*. The analysis is based on [43].

Several topics concerning primal-dual logarithmic barrier methods are treated in Chapter 6. In the first instance, the analysis of Jiang [57] is refined and extended. This leads to results on quadratic convergence to the central path, as well as a weaker condition which allows a full Newton step. The analysis is used to formulate full Newton step methods which use adaptive (dynamic) updates of the barrier (centering) parameter μ. These methods were first described by De Klerk *et al.* in [26]. The tools developed for the analysis are further used to give a simplified complexity analysis of the potential reduction method by Nesterov and Todd [85], and the long step method by Jiang [57].

Chapter 2

Initialization via self–dual embeddings

A comprehensive treatment of a specific initialization strategy is presented here, namely self-dual embedding; the original primal and dual problems are embedded in a larger problem with a known interior feasible starting point. A framework for infeasible–start algorithms with the best obtainable complexity bound is thus presented. The main results concern embedding extended Lagrange-Slater dual (ELSD) problems (as opposed to Lagrangian duals), in order to detect general infeasibility. Remaining difficulties are stated clearly, and two open problems are posed.

2.1 Introduction

Many SDP algorithms found in the literature require feasible starting points. So-called 'big-M' methods (see *e.g.* [112]) are often employed in practice to obtain feasible starting points.

In the LP case an elegant solution for the initialization problem is to embed the original problem in a skew–symmetric self–dual problem which has a known interior feasible solution on the central path [120, 47]. The solution of the embedding problem then yields the optimal solution to the original problem, or gives a certificate of either infeasibility or unboundedness. In this way detailed information

about the solution is obtained. The idea of self-dual embeddings for LP dates back to the 1950's and the work of Goldman and Tucker [38]. With the arrival of interior point methods, the embedding idea was revived to be used in infeasible–start algorithms.

In spite of the desirable theoretical properties of self–dual embeddings, the idea did not receive immediate recognition in implementations, due to the fact that the embedding problem has a dense column in the coefficient matrix. This can lead to fill-in of Choleski factorizations during computation.

Despite this perception, Xu et al. [115] have made a successful implementation for LP using the embedding, and it has even been implemented as an option in the well-known commercial LP solver CPLEX-barrier [18], and most recently in the APOS solver [6]. The common consensus now is that this strategy promises to be competitive in practice [7] (see also [34] and [98]).

A homogeneous embedding of monotone nonlinear complementarity problems is discussed by Andersen and Ye in [8]. For semidefinite programming the homogeneous embedding idea was first developed by Potra and Sheng [89]. The embedding strategy was extended by De Klerk *et al.* in [22] and independently by Luo *et al.* [69] to obtain self-dual embedding problems with nonempty interiors. The resulting embedding problem has a known centered starting point, unlike the homogeneous embedding; it can therefore be solved using any feasible path-following interior point method. This is an advantage in the SDP case, where many possible primal-dual algorithms are available, while none has yet emerged as clear favourite.

A so–called *maximally complementary* solution (*e.g.* the limit of the central path) of the embedding problem yields one of the following alternatives about the original problem pair:

(I) an optimal solution with zero duality gap for the original problem is obtained;

(II) an improving ray is obtained for either the primal and/or dual problem (so-called *strong infeasibility* is detected);

(III) a certificate is obtained that no optimal solution pair with zero duality gap exists and that neither the primal nor the dual problem has an improving ray. This can only happen if one or both of the primal and dual SDP problems fail to satisfy the Slater regularity condition.

Loosely speaking, the original primal and dual problems are solved if a complementary solution pair exists, or if one or both of the problems are strongly infeasible.

Unfortunately, some pathological duality effects can occur for SDP[1] which are absent from LP, for example:

- A positive duality gap at an optimal primal-dual solution pair;
- an arbitrarily small duality gap can be attained by feasible primal-dual pairs, but no optimal pair exists;
- an SDP problem may have an optimal solution even though its (Lagrangian) dual is infeasible.

In cases like these little or no information could be given in [22]. The embedding approach was therefore further extended by De Klerk *et al.* [24] in order to answer the following questions:

(1) How can interior point algorithms be employed to solve the embedding problem to ϵ-optimality?

(2) How can one decide which variables are zero in a maximally complementary solution of the embedding problem, if only an ϵ-optimal solution is known? (This is important in drawing conclusions about the original problem pair from an ϵ-optimal solution of the embedding problem.)

(3) How is infeasibility and unboundedness detected in general? (This was done in [24] by using extended Lagrange-Slater dual problems [92] in the embedding, where necessary. In this way the optimal value of a given SDP problem can be obtained if it is finite.[2])

In this chapter the combined results of De Klerk *et al.* [22] and [24] are presented and elaborated on.

[1] Examples of these effects will be given in Sections 2.2 and 2.7, and can also be found in [112] and [69].

[2] The problem of how to use the extended Lagrange-Slater dual problems in infeasible-start algorithms was posed by Ramana and Pardalos in [95].

Outline of the chapter

After some preliminaries, a review of recent results concerning the convergence of the central path is given in Section 2.3, with simplified proofs. The embedding strategy is discussed thereafter in Section 2.4. Solution strategies for the embedding problem are given in Section 2.5. In Section 2.6 it is shown how to interpret an ϵ-optimal solution of the embedding problem in order to draw conclusions about the solution of the original problem pair (*i.e.* to distinguish between the abovementioned cases (I) to (III)). The remaining difficulties are highlighted. Remaining duality issues and ways of detecting weak infeasibility are discussed in Section 2.7. In Section 2.8 it is shown how extended Lagrange-Slater duals can be used in the embedding strategy instead of Lagrangian duals to give a certificate of the status of a given problem. Finally, some conclusions are drawn and extensions to more general convex optimization problems are discussed.

2.2 Preliminaries

2.2.1 Feasibility issues

We say that a problem (P) and its Lagrangian dual (D) are in *standard form* if they are in the form

$$(P): \quad p^* := \inf_X \{\mathbf{Tr}\,(CX) \ : \mathbf{Tr}\,(A_i X) = b_i \ (i = 1, \ldots, m),\ X \succeq 0\}\,,$$

and

$$(D): \quad d^* := \sup_{y,S} \left\{ b^T y \ : \ \sum_{i=1}^m y_i A_i + S = C,\ S \succeq 0,\ y \in \mathbb{R}^m \right\}.$$

The solutions X and (y, S) will be referred to as *feasible solutions* as they satisfy the primal and dual constraints respectively.

The values p^* and d^* will be called the optimal values of (P) and (D), respectively. We use the convention that $p^* = -\infty$ if (P) is unbounded and $p^* = \infty$ if (P) is infeasible, with the analogous convention for (D). The primal and dual feasible sets will be denoted by $\mathcal{P}$ and $\mathcal{D}$ respectively, and $\mathcal{P}^*$ and $\mathcal{D}^*$ will denote the respective optimal sets, *i.e.*

$$\mathcal{P}^* = \{X \in \mathcal{P} \ : \ \mathbf{Tr}\,(CX) = p^*\} \text{ and } \mathcal{D}^* = \left\{(S, y) \in \mathcal{D} \ : \ b^T y = d^*\right\}.$$

A problem (P) (resp. (D)) is called solvable if $\mathcal{P}^*$ (resp. $\mathcal{D}^*$) is nonempty.

The following assumptions will be used throughout, unless otherwise mentioned:

A1: the matrices A_i $(i = 1, \ldots, m)$ are linearly independent;

A2: the feasible sets $\mathcal{P}$ and $\mathcal{D}$ have nonempty interiors.

Under assumption **A1**, y is uniquely determined for a given dual feasible S. This is the same assumption as the assumption in LP that the constraint matrix must have full rank. To see this, note that the linear indepedence of A_i $(i = 1, \ldots, m)$, is equivalent to the linear independence of $\mathbf{vec}(A_i)$ $(i = 1, \ldots, m)$. Practical algorithms for ensuring full row rank of a matrix are described in [5].

Assumption **A2** is called *strict feasibility*.

Definition 2.2.1 (Strict feasibility) *A problem (P) (resp. (D)) is called strictly feasible if there exists $X \in \mathcal{P}$ with $X \succ 0$ (resp. $(y, S) \in \mathcal{D}$ with $S \succ 0$).*

We will see in this chapter that assumption **A2** can (almost) be made without loss of generality, since a given SDP problem can be embedded in a strictly feasible self-dual problem. The embedding problem can then be solved instead of the original problem, by using the algorithms described in the next four chapters.

Strict feasibility is equivalent to the well-known *Slater's constraint qualification* or *Slater's regularity condition*. This follows from the fact that that the interior of the cone of positive semidefinite matrices consists of the positive definite matrices.

To decide about possible infeasibility and unboundedness of the problems (P) and (D) we need the following definition.

Definition 2.2.2 *We say that the primal problem (P) has an improving ray if there is a symmetric matrix $\bar{X} \succeq 0$ such that $\mathbf{Tr}\,(A_i\bar{X}) = 0, \;\; \forall\, i$ and $\mathbf{Tr}\,(C\bar{X}) < 0$.*

Analogously, the dual problem (D) has an improving ray if there is a vector $\bar{y} \in \mathbb{R}^m$ such that $\bar{S} := -\sum_{i=1}^m \bar{y}_i A_i \succeq 0$ and $b^T\bar{y} > 0$.

Primal improving rays cause infeasibility of the dual problem, and *vice versa*. Formally one has the following result.

Lemma 2.2.1 *If there is a dual improving ray $\bar{y}$ then (P) is infeasible. Similarly, a primal improving ray $\bar{X}$ implies infeasibility of (D).*

Proof:

Let a dual improving ray $\bar{y}$ be given. By assuming the existence of a primal feasible X one has

$$0 < b^T \bar{y} = \sum_{i=1}^{m} \bar{y}_i \mathbf{Tr}\left(A_i X\right) = -\mathbf{Tr}\left(X\bar{S}\right) \leq 0,$$

which is a contradiction. The proof in case of a primal improving ray proceeds similarly. □

Definition 2.2.3 *Problem (P) (resp. (D)) is called strongly infeasible if (D) (resp. (P)) has an improving ray.*

Every infeasible LP problem is strongly infeasible, but in the SDP case so-called *weak infeasibility* is also possible.

Definition 2.2.4 *Problem (P) is weakly infeasible if $\mathcal{P} = \emptyset$ and for each $\epsilon > 0$ there exists an $X \succeq 0$ such that*

$$|\mathbf{Tr}\left(A_i X\right) - b_i| \leq \epsilon, \quad \forall i.$$

Similarly, problem (D) is called weakly infeasible if $\mathcal{D} = \emptyset$ and for every $\epsilon > 0$ there exist $y \in \mathbb{R}^m$ and $S \succeq 0$ such that

$$\left\| \sum_{i=1}^{m} y_i A_i + S - C \right\| \leq \epsilon.$$

Example 2.2.1 *An example of weak infeasibility is given if (D) is defined by: find*

$$\sup y_1$$

subject to

$$y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where we can construct an 'ϵ-infeasible solution' by setting

$$S = \begin{bmatrix} 1/\epsilon & 1 \\ 1 & \epsilon \end{bmatrix}, \; y_1 = -\frac{1}{\epsilon}.$$

□

It can be shown [69] that an infeasible SDP problem is either weakly infeasible or strongly infeasible.

2.2.2 Optimality and complementarity

Recall that the duality gap for (P) and (D) at solutions $X \in \mathcal{P}$ and $(y, S) \in \mathcal{D}$ is given by

$$\mathbf{Tr}\,(CX) - b^T y = \mathbf{Tr}\,\left((\sum_{i=1}^{m} y_i A_i + S)X\right) - \sum_{i=1}^{m} y_i \mathbf{Tr}\,(A_i X) = \mathbf{Tr}\,(SX).$$

This shows that the duality gap is always nonnegative, since $X \succeq 0$ and $S \succeq 0$. This is the well-known *weak duality* property. The optimal duality gap is said to be zero if

$$p^* \equiv \inf_{X \in \mathcal{P}} \mathbf{Tr}\,(CX) = \sup_{S,y \in \mathcal{D}} b^T y \equiv d^*. \tag{2.1}$$

Note that this definition does not imply that $\mathcal{P}^*$ and $\mathcal{D}^*$ are nonempty.

It is well-known from convex programming (see *e.g.* [12]) that if both (P) and (D) are strictly feasible, then $\mathcal{P}^*$ and $\mathcal{D}^*$ are nonempty and the duality gap is zero, *i.e.* strong duality holds. If both (P) and (D) are feasible, and one is strictly feasible, then (2.1) is also guaranteed to hold.

The optimality conditions for (P) and (D) are

$$\left.\begin{array}{rcl} \mathbf{Tr}\,(A_i X) & = & b_i, \quad X \succeq 0, \quad i = 1, \ldots, m \\ \sum_{i=1}^{m} y_i A_i + S & = & C, \quad S \succeq 0 \\ XS & = & 0. \end{array}\right\} \tag{2.2}$$

Feasible solutions X and S satisfying the last equality constraint are called *complementary*. Since X and S are symmetric positive semi–definite matrices the complementarity of X and S $(XS = 0)$ is equivalent to $\mathbf{Tr}\,(XS) = 0$. Complementary feasible solutions therefore are optimal with zero duality gap.

It will be convenient to introduce subspaces $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{T}$ of $\mathbb{R}^n$ as follows: $\mathcal{B}$ is the subspace generated by all columns occurring in primal optimal solutions X,

$\mathcal{N}$ the subspace generated by all columns occurring in dual optimal solutions S and $\mathcal{T}$ the orthogonal complement of the subspace $\mathcal{B} + \mathcal{N}$. For any primal-dual optimal pair (X, S) we have $XS = 0$, implying that the column spaces of X and S are orthogonal. Consequently, the subspaces $\mathcal{B}$ and $\mathcal{N}$ are orthogonal as well. Thus the subspaces $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{T}$ partition $\mathbb{R}^n$ into three mutually orthogonal spaces.

The range (or column) space of any primal (resp. dual) feasible X (S) is denoted as $\mathcal{R}(X)$ (resp. $\mathcal{R}(S)$). If X is primal optimal and $\mathcal{R}(X) = \mathcal{B}$ then we call X a *maximally complementary primal solution* and, similarly, if S is dual optimal and $\mathcal{R}(S) = \mathcal{N}$ then S is called a *maximally complementary dual solution*. If X and S are both maximally complementary then we call (X, S) a *maximally complementary optimal pair*; if moreover $\mathcal{T} = 0$ we call the pair (X, S) *strictly complementary*.

In the following sections it will become clear that maximally complementary solutions exist.[3] Before proceeding we introduce some more notation. Since $X \succeq 0$ and $S \succeq 0$ and optimal X and S commute ($XS = SX = 0$), the spectral (eigenvector-eigenvalue) decompositions of an optimal pair X and S take the form:

$$X = Q\Lambda Q^T, \quad S = Q\Sigma Q^T, \tag{2.3}$$

where Q is orthogonal and the diagonal matrices Λ and Σ have the (nonnegative) eigenvalues of X and S on their respective diagonals. Obviously $XS = 0$ if and only if $\Lambda\Sigma = 0$. Furthermore, $\mathcal{R}(X) = \mathcal{R}(Q\Lambda)$ and $\mathcal{R}(S) = \mathcal{R}(Q\Sigma)$.

2.2.3 Orthogonality

The well-known *orthogonality property* is trivially proven, and will be used extensively.

Lemma 2.2.2 (Orthogonality) *Let (X, S) and (X^0, S^0) be two pairs of feasible solutions. Denoting $\Delta X = X - X^0$ and $\Delta S = S - S^0$, one has:*

$$\mathbf{Tr}\ (\Delta X \Delta S) = 0.$$

[3] Results pertaining to bounds on the rank of optimal solutions may be found in [87, 88], and on nondegeneracy and strict complementarity properties of optimal solutions in [3].

2.3 Features of the central path

In this section we assume that (P) and (D) are strictly feasible. The analysis of this section will then apply to the embedding problem presented in the next section, as the embedding problem will be self-dual and strictly feasible.

Recall that if the optimality conditions (2.2) for (P) and (D) are relaxed to

$$\left.\begin{array}{rcll} \mathbf{Tr}\,(A_i X) & = & b_i, \quad X \succeq 0, & i = 1, \ldots, m \\ \sum y_i A_i + S & = & C, \quad S \succeq 0 & \\ XS & = & \mu I, & \end{array}\right\} \tag{2.4}$$

with $\mu > 0$, then this system has a unique solution, denoted by $X(\mu), S(\mu), y(\mu)$. This solution can be seen as the parametric representation of a smooth curve (the *central path*) in terms of the parameter μ. The existence and uniqueness of the central path can be proved in the following way: Consider the problem

$$\min_X \{ f_p(X) := \mathbf{Tr}\,(CX) - \mu \log\det X \;:\; \mathbf{Tr}\,(A_i X) = b_i \;(i = 1, \ldots, m),\; X \succ 0 \}$$

i.e. the minimization of the *primal log-barrier function* over the interior of $\mathcal{P}$. The KKT conditions for this problem are[4]:

$$\begin{array}{rcll} \nabla f_p(X) = C - \mu X^{-1} & = & \displaystyle\sum_{i=1}^{m} y_i A_i & \\ \mathbf{Tr}\,(A_i X) & = & b_i & i = 1, \ldots, m \\ X & \succ & 0. & \end{array}$$

Defining $S = C - \sum_{i=1}^{m} y_i A_i$ this system becomes identical to system (2.4). In other words, the existence and uniqueness of the central path is equivalent to the existence of a unique minimizer of f_p in $\mathrm{int}\,(\mathcal{P})$ for each $\mu > 0$. The function f_p is strictly convex, and any minimizer of f_p is therefore unique. Existence of a minimizer may be proved by showing that the level sets of f_p are compact if (P) and (D) are strictly feasible.

In this section we will show that any sequence along the central path has accumulation points in the optimal set, and that these accumulation points are maximally complementary. Then we will prove that as $\mu \to 0$ the central path converges to a

[4]The gradient of f_p is derived in Appendix A.

maximally complementary solution pair. The limit is the so-called *analytic center* of the optimal set which will be defined later on.

In what follows we consider a fixed sequence $\{\mu_t\} \rightarrow 0$ with $\mu_t > 0$, $t = 1, \cdots$, and prove that there exists a subsequence of $\{X(\mu_t), S(\mu_t)\}$ which converges to a maximally complementary solution. The existence of limit points of the sequence is an easy consequence of the following lemma.

Lemma 2.3.1 *Given $\bar{\mu} > 0$, the set*

$$\{(X(\mu), S(\mu)) \ : \ 0 < \mu \leq \bar{\mu}\}$$

is bounded.

Proof:

Let (X^0, S^0) be any strictly feasible primal-dual solution, and $(X(\mu), S(\mu))$ a central solution corresponding to some $\mu > 0$. By orthogonality, Lemma 2.2.2, one has

$$\mathbf{Tr}\ \Big((X(\mu) - X^0)(S(\mu) - S^0)\Big) = 0. \tag{2.5}$$

The centrality conditions imply $\mathbf{Tr}\ (X(\mu)S(\mu)) = n\mu$, which simplifies (2.5) to

$$\mathbf{Tr}\,(X(\mu)S^0) + \mathbf{Tr}\,(X^0S(\mu)) = n\mu + \mathbf{Tr}\,(X^0S^0). \tag{2.6}$$

The left hand side terms of the last inequality are nonnegative by feasibility. One therefore has

$$\mathbf{Tr}\ \Big(X(\mu)S^0\Big) \leq n\mu + \mathbf{Tr}\,(X^0S^0),$$

which for a given $\bar{\mu} > 0$ implies

$$\mathbf{Tr}\ (X(\mu)) \leq \frac{n\bar{\mu} + \mathbf{Tr}\,(X^0S^0)}{\lambda_{\min}(S^0)}, \quad \forall\ \mu \leq \bar{\mu}$$

where $\lambda_{\min}(S^0)$ denotes the smallest eigenvalue of S^0. Now using the fact that any positive semidefinite matrix X satisfies $\|X\| \leq \mathbf{Tr}\,(X)$ for the Frobenius norm, one has

$$\|X(\mu)\| \leq \frac{n\bar{\mu} + \mathbf{Tr}\,(X^0S^0)}{\lambda_{\min}(S^0)}, \quad \forall\ \mu \leq \bar{\mu}.$$

A similar bound can be derived for $\|S(\mu)\|$. □

Now let

$$X(\mu_t) := Q(\mu_t)\Lambda(\mu_t)Q(\mu_t)^T, \quad S(\mu_t) := Q(\mu_t)\Sigma(\mu_t)Q(\mu_t)^T$$

denote the spectral (eigenvector-eigenvalue) decompositions of $X(\mu_t)$ and $S(\mu_t)$. Lemma 2.3.1 implies that the eigenvalues of $X(\mu_t)$ and $S(\mu_t)$ are bounded. The matrices $Q(\mu_t)$ are orthonormal for all t, and are therefore likewise restricted to a compact set. It follows that the sequence of triples $(Q(\mu_t), \Lambda(\mu_t), \Sigma(\mu_t))$ has an accumulation point, $(Q^*, \Lambda^*, \Sigma^*)$ say. Thus there exists a subsequence of $\{\mu_t\}$ (still denoted by $\{\mu_t\}$ for the sake of simplicity) such that

$$\lim_{t\to\infty} Q(\mu_t) = Q^* \quad \lim_{t\to\infty} \Lambda(\mu_t) = \Lambda^*, \quad \lim_{t\to\infty} \Sigma(\mu_t) = \Sigma^*.$$

Note that $\Lambda(\mu_t)\Sigma(\mu_t) = \mu I$. Thus, defining

$$X^* := Q^*\Lambda^*Q^{*T} = \lim_{t\to\infty} X(\mu_t), \quad S^* := Q^*\Sigma^*Q^{*T} = \lim_{t\to\infty} S(\mu_t), \tag{2.7}$$

we have $\Lambda^*\Sigma^* = 0$ and the pair (X^*, S^*) is optimal.

Theorem 2.3.1 (Maximal complementarity) *The pair (X^*, S^*) as defined in (2.7) is maximally complementary.*

Proof:

Let (X, S) be an arbitrary optimal pair. Applying the orthogonality property (Lemma 2.2.2) and $\mathbf{Tr}\,(XS) = 0$, $\mathbf{Tr}\,(X(\mu_t)S(\mu_t)) = n\mu_t$ we obtain

$$\mathbf{Tr}\,(X(\mu_t)S) + \mathbf{Tr}\,(XS(\mu_t)) = n\mu_t.$$

Since $X(\mu_t)S(\mu_t) = \mu_t I$, dividing both sides by μ_t yields

$$\mathbf{Tr}\,(S(\mu_t)^{-1}S) + \mathbf{Tr}\,(XX(\mu_t)^{-1}) = n \tag{2.8}$$

for all t. This implies

$$\mathbf{Tr}\,(XX(\mu_t)^{-1}) \le n, \quad \mathbf{Tr}\,(S(\mu_t)^{-1}S) \le n, \tag{2.9}$$

since both terms in the left hand side of (2.8) are nonnegative. We derive from this that X^* and S^* are maximally complementary. Below we give the derivation for X^*; the derivation for S^* is similar and is therefore omitted.

Denoting the i-th column of the orthonormal (eigenvector) matrix $Q(\mu_t)$ by $q_i(\mu_t)$ and the i-th diagonal element of the (eigenvalue) matrix $\Lambda(\mu_t)$ by $\lambda_i(\mu_t)$ we have

$$X(\mu_t)^{-1} = Q(\mu_t)\Lambda(\mu_t)^{-1}Q(\mu_t)^T = \sum_{i=1}^{n} \frac{1}{\lambda_i(\mu_t)} q_i(\mu_t)q_i(\mu_t)^T. \tag{2.10}$$

Substituting (2.10) into the first inequality in (2.9) yields

$$\mathbf{Tr}\left(XX(\mu_t)^{-1}\right) = \sum_{i=1}^{n} \mathbf{Tr}\left(\frac{1}{\lambda_i(\mu_t)} Xq_i(\mu_t)q_i(\mu_t)^T\right) = \sum_{i=1}^{n} \frac{q_i(\mu_t)^T X q_i(\mu_t)}{\lambda_i(\mu_t)} \leq n. \tag{2.11}$$

The last inequality implies

$$q_i(\mu_t)^T X q_i(\mu_t) \leq n\lambda_i(\mu_t), \quad i = 1, 2, \cdots, n.$$

Letting t go to infinity we obtain

$$q_i^{*T} X q_i^* \leq n\lambda_i^*, \quad i = 1, 2, \cdots, n,$$

where q_i^* denotes the i-th column of Q^* and λ_i^* the i-th diagonal element of Λ^*. Thus we have $q_i^{*T} X q_i^* = 0$ whenever $\lambda_i^* = 0$. This implies

$$Xq_i^* = 0 \text{ if } \lambda_i^* = 0, \tag{2.12}$$

since $(q_i^*)^T X q_i^* = \left\| X^{\frac{1}{2}} q_i^* \right\|^2$, where $X^{\frac{1}{2}}$ is the symmetric square root factor of X. In other words, the row space of X is orthogonal to each column q_i^* of Q^* for which $\lambda_i^* = 0$. Hence the row space of X is a subspace of the space generated by the columns q_i^* of Q^* for which $\lambda_i^* > 0$. The latter space is just $\mathcal{R}(Q^*\Lambda^*)$ which equals $\mathcal{R}(X^*)$. Since X is symmetric we conclude that $\mathcal{R}(X) \subseteq \mathcal{R}(X^*)$. Since X is an arbitrary primal optimal solution, this implies that $\mathcal{R}(X^*) = \mathcal{B}$, and hence the proof is complete. □

Notation: In what follows we define

$$\begin{aligned} B &:= \{i \,:\, \lambda_i^* > 0\}, \\ N &:= \{i \,:\, \sigma_i^* > 0\}, \\ T &:= \{1, 2, \cdots, n\} \setminus (B \cup N). \end{aligned}$$

Then the sets B, N and T form a partition of the full index set $\{1, 2, \cdots, n\}$. Let Q_J^* denote the submatrix of Q^* consisting of the columns indexed by $J \subseteq \{1, 2, \cdots, n\}$. (The matrices $Q_J(\mu)$ and $\Lambda_J(\mu)$ are defined similarly.)

Using this notation, it follows from Theorem 2.3.1 that any optimal pair (X, S) can be written as

$$X = Q_B^* U_X {Q_B^*}^T, \text{ and } S = Q_N^* U_S {Q_N^*}^T \tag{2.13}$$

for suitable matrices U_X and U_S. In fact, since ${Q_B^*}^T Q_B^*$ is equal to the identity matrix I_B of size $|B|$ and ${Q_N^*}^T Q_N^*$ equals the identity matrix I_N of size $|N|$, U_X and U_S uniquely follow from

$$U_X = {Q_B^*}^T X Q_B^*, \text{ and } U_S = {Q_N^*}^T S Q_N^*. \tag{2.14}$$

Note that U_X and U_S are symmetric. It can easily be understood that the matrices X and U_X have the same spectrum, except that the multiplicity of zero in the spectrum of X will be larger than in the spectrum of U_X. Note that U_{X^*} is just the minor of Λ^* determined by the indices in B. Hence the eigenvalues of U_{X^*} are all positive, and therefore $\det(U_{X^*}) > 0$.

Definition 2.3.1 (Analytic center) *The analytic center of $\mathcal{P}^*$ is the (unique) solution of the maximization problem*

$$\max_{X \in \mathcal{P}^*} \log\det(U_X).$$

Similarly, the analytic center of $\mathcal{D}^*$ is the (unique) solution of the maximization problem

$$\max_{S \in \mathcal{D}^*} \log\det(U_S).$$

The uniqueness of the analytic centers follows from the strict convexity of the function $f_{bar}(X) = -\log\det(X)$ and the compactness of the optimal sets $\mathcal{P}^*$ and $\mathcal{D}^*$. Note that the analytic center is necessarily a maximally complementary solution. We now prove the convergence of the central path to the analytic center of the optimal set under the assumption that a strictly complementary solution exists (*i.e.* $T = \emptyset$, or, equivalently $\mathcal{T} = \{0\}$). This result has been proved by Ye [117] for general self-scaled conic problems. It is nevertheless insightful to derive the proof for the semidefinite case, which is analogous to the proof in the LP case. The assumption of strict complementarity simplifies things, but is not necessary

— Goldfarb and Scheinberg [37] have proved the result in the general case where no strictly complementary solution is available; a proof of the general case is also included here, but will follow the proof where strict complementarity is assumed.

Theorem 2.3.2 *If $\mathcal{T} = \{0\}$ then X^* is the analytic center of $\mathcal{P}^*$ and S^* is the analytic center of $\mathcal{D}^*$.*

Proof:

Just as in the proof of Theorem 2.3.1, let (X, S) be an arbitrary optimal pair. We may rewrite (2.8) as

$$\sum_{i=1}^{n} \frac{q_i(\mu_t)^T X q_i(\mu_t)}{\lambda_i(\mu_t)} + \sum_{i=1}^{n} \frac{q_i(\mu_t)^T S q_i(\mu_t)}{\sigma_i(\mu_t)} = n.$$

Since all terms in the above sums are nonnegative, this implies

$$\sum_{i \in B} \frac{q_i(\mu_t)^T X q_i(\mu_t)}{\lambda_i(\mu_t)} + \sum_{i \in N} \frac{q_i(\mu_t)^T S q_i(\mu_t)}{\sigma_i(\mu_t)} \leq n.$$

Letting t go to infinity we obtain

$$\sum_{i \in B} \frac{q_i^{*T} X q_i^*}{\lambda_i^*} + \sum_{i \in N} \frac{q_i^{*T} S q_i^*}{\sigma_i^*} \leq n.$$

This can be rewritten as

$$\mathbf{Tr}\ \left(X Q_B^* U_{X^*}^{-1} {Q_B^*}^T\right) + \mathbf{Tr}\ \left(S Q_N^* U_{S^*}^{-1} {Q_N^*}^T\right) \leq n,$$

or

$$\mathbf{Tr}\ \left({Q_B^*}^T X Q_B^* U_{X^*}^{-1}\right) + \mathbf{Tr}\ \left({Q_N^*}^T S Q_N^* U_{S^*}^{-1}\right) \leq n.$$

Using the definition of U_X and U_S in (2.14), this implies

$$\mathbf{Tr}\ \left(U_X U_{X^*}^{-1}\right) + \mathbf{Tr}\ \left(U_S U_{S^*}^{-1}\right) \leq n.$$

Since $T = \emptyset$, we have $|B| + |N| = n$. Recall that the matrix $U_X U_{X^*}^{-1}$ has size $|B| \times |B|$ and $U_S U_{S^*}^{-1}$ has size $|N| \times |N|$. Applying the arithmetic-geometric mean inequality to the eigenvalues of these matrices we get

$$\det\left(U_X U_{X^*}^{-1}\right) \det\left(U_S U_{S^*}^{-1}\right) \leq \left(\frac{1}{n}\left(\mathbf{Tr}\ \left(U_X U_{X^*}^{-1}\right) + \mathbf{Tr}\ \left(U_S U_{S^*}^{-1}\right)\right)\right)^n \leq 1,$$

which implies

$$\det(U_X)\det(U_S) \leq \det(U_{X^*})\det(U_{S^*}). \tag{2.15}$$

Substituting $S = S^*$ in (2.15) gives $\det(U_X) \leq \det(U_{X^*})$ and by setting $X = X^*$ we obtain $\det(U_S) \leq \det(U_{S^*})$. Thus we have shown that X^* is the analytic center of $\mathcal{P}^*$ and S^* the analytic center of $\mathcal{D}^*$. □

In order to relax the assumption of $\mathcal{T} = \{0\}$, we first show that the central path passes through the analytic centers of the level sets $\mathbf{Tr}(XS) = n\mu$.

Lemma 2.3.2 *Let $X \in \mathcal{P}$ and $S \in \mathcal{D}$ satisfy $\mathbf{Tr}(XS) = n\mu$. One has*

$$\det(XS) \leq \det(X(\mu)S(\mu)),$$

i.e. *the centered pair $(X(\mu), S(\mu)$ is the analytic center of the level set*

$$\{(X,S) \ : \ \mathbf{Tr}(XS) = n\mu, \ X \in \mathcal{P}, \ S \in \mathcal{D}\}.$$

Proof:

By orthogonality one has

$$\mathbf{Tr}\ (X(\mu) - X)(S(\mu) - S) = 0,$$

as before. Using $X(\mu)S(\mu) = \mu I$ and $\mathbf{Tr}(XS) = n\mu$, simplifies this to

$$\mathbf{Tr}\ \left(X(\mu)^{-1}X\right) + \mathbf{Tr}\ \left(S(\mu)^{-1}S\right) = 2n.$$

Applying the arithmetic-geometric inequality to the eigenvalues of $X(\mu)^{-1}X$ and $S(\mu)^{-1}S$ yields

$$\left[\det\left(X(\mu)^{-1}XS(\mu)^{-1}S\right)\right]^{\frac{1}{2n}} \leq \frac{1}{2n}\left[\mathbf{Tr}\ \left(X(\mu)^{-1}X\right) + \mathbf{Tr}\ \left(S(\mu)^{-1}S\right)\right] = 1,$$

which implies the required result. □

The result of the lemma already makes it plausible that the central path converges to the analytic center of the optimal set, even if no strictly complementary solution

exists. In order to give a rigorous proof of this fact, note that we can separate the problem:

$$(X(\mu), S(\mu)) = \arg\max_{X,S} \{\log\det XS \ : \ X \in \mathcal{P}, \ S \in \mathcal{D}, \ \mathbf{Tr}\,(XS) = n\mu\}$$

into primal and dual problems

$$X(\mu) = \arg\max_{X} \{\log\det X \ : \ X \in \mathcal{P}, \ \mathbf{Tr}\,(CX) = p(\mu)\} \tag{2.16}$$

and

$$S(\mu) = \arg\max_{y,S} \left\{\log\det S \ : \ (y,S) \in \mathcal{D}, \ b^T y = d(\mu)\right\}, \tag{2.17}$$

where $p(\mu)$ and $d(\mu)$ denote the objective values of the primal and dual μ-centers respectively.

We proceed to show that $S(\mu_t)$ converges to the analytic center of $\mathcal{D}^*$, and subsequently indicate how to do the analogous analysis for $X(\mu_t)$. The method of proof is essentially that of Goldfarb and Scheinberg [37], where the proof was done for the primal problem.

Theorem 2.3.3 (Convergence of the dual central path) *The sequence* $\{y(\mu_t), S(\mu_t)\}$ *converges to the analytic center of* $\mathcal{D}^*$.

Proof:

We know from Lemma 2.3.2 that $y(\mu_t), S(\mu_t)$ satisfy the KKT optimality conditions of problem (2.17). Recall that the spectral decomposition of $S(\mu_t)$ may be written as

$$\begin{aligned} S(\mu_t) &\equiv Q_N(\mu_t)\Lambda_N(\mu_t)Q_N(\mu_t)^T + Q_{B\cup T}(\mu_t)\Lambda_{B\cup T}(\mu_t)Q_{B\cup T}(\mu_t)^T \\ &= [Q_N(\mu_t),\ Q_{B\cup T}(\mu_t)] \begin{bmatrix} \Lambda_N(\mu_t) & 0 \\ 0 & \Lambda_{B\cup T}(\mu_t) \end{bmatrix} [Q_N(\mu_t),\ Q_{B\cup T}(\mu_t)]^T, \end{aligned}$$

where $Q_N(\mu_t) \to Q_N^*$, $Q_{B\cup T}(\mu_t) \to Q_{B\cup T}^*$, $\Lambda_{B\cup T}(\mu_t) \to 0$, and $\Lambda_N(\mu_t) \to U_{S^*} \succ 0$. The solution of problem (2.17) therefore remains unchanged if we set $y = y(\mu_t)$ and further require the variable S to be of the form

$$S = [Q_N(\mu_t),\ Q_{B\cup T}(\mu_t)] \begin{bmatrix} U_S & 0 \\ 0 & \Lambda_{B\cup T}(\mu_t) \end{bmatrix} [Q_B(\mu_t),\ Q_{B\cup T}(\mu_t)]^T,$$

where everything is now fixed except $U_S \succ 0$. Substitution of the expression for S changes problem (2.17) to

$$\max_{U_S} \log\det U_S$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} y_i(\mu_t)A_i + Q_N(\mu_t)U_S Q_N(\mu_t)^T + Q_{B\cup T}(\mu_t)\Lambda_{B\cup T}(\mu_t)Q_{B\cup T}(\mu_t)^T &= C \\ U_S &\succ 0. \end{aligned}$$

This problem has optimality conditions

$$\left.\begin{aligned} U_S^{-1} - Q_N(\mu_t)^T Z(\mu_t) Q_N(\mu_t) &= 0 \\ \textstyle\sum_{i=1}^{m} y_i(\mu_t)A_i + Q_N(\mu_t)U_S Q_N(\mu_t)^T + Q_{B\cup T}(\mu_t)\Lambda_{B\cup T}(\mu_t)Q_{B\cup T}(\mu_t)^T &= C \\ U_S \succ 0, \quad Z(\mu_t) \succeq 0 & \end{aligned}\right\} \tag{2.18}$$

for some Lagrange multiplier $Z(\mu_t) \in \mathcal{S}_n^+$. Now recall that $U_S = \Lambda_N(\mu_t)$ is the unique solution of the optimality conditions (2.18) and consequently we can choose

$$Z(\mu_t) = Q_N(\mu_t)\Lambda_N(\mu_t)^{-1}Q_N(\mu_t)^T. \tag{2.19}$$

Note that $\Lambda_N(\mu_t)^{-1} \to U_{S^*}^{-1}$, and we can therefore assume that $Z(\mu_t) \to Z \succeq 0$, say.

Taking the limit as $\mu_t \to 0$ of the system of optimality conditions yields

$$\begin{aligned} {U_S}^{-1} - {Q_N^*}^T Z Q_N^* &= 0 \\ \sum_{i=1}^{m} y_i A_i + Q_N^* U_S {Q_N^*}^T &= C \\ U_S \succ 0, \quad Z \succeq 0. & \end{aligned}$$

These equations are the optimality conditions of the problem which defines the analytic center of $\mathcal{D}^*$, namely

$$\max_{U_S} \left\{ \log\det U_S \ : \ Q_N^* U_S {Q_N^*}^T \in \mathcal{D}^* \right\}.$$

□

The convergence of the primal central path now follows by reformulating (P) and (D) in the *conic formulation*. To this end, let $M \in \mathcal{S}_n$ be such that $\mathbf{Tr}\,(A_i M) = b_i$ $(i = 1, \ldots, m)$, and let $L = \text{span}\,\{A_1, \ldots, A_m\}$. Then (P) has the alternative formulation

$$\min_X \left\{ \mathbf{Tr}\,(CX) \;:\; X \in L^\perp + M,\; X \succeq 0 \right\},$$

and the Lagrangian dual of this problem is the conic reformulation of (D):

$$\max_S \left\{ \mathbf{Tr}\,(MS) \;:\; S \in L + C,\; S \succeq 0 \right\}.$$

Thus problems (P) and (D) have been cast in the same form, and therefore Theorem 2.3.3 can be used to establish the convergence of $X(\mu)$ to the analytic center of $\mathcal{P}^*$.

2.4 The embedding strategy

In the remainder of this chapter, no assumptions are made about feasibility of (P) and (D).

Consider the following *homogeneous embedding* of (P) and (D):

$$\left.\begin{array}{lllllll} & \mathbf{Tr}\,(A_i X) & -\tau b_i & & & = 0 & \forall i \\ -\sum_{i=1}^m y_i A_i & & +\tau C & -S & & = 0 & \\ b^T y & -\mathbf{Tr}\,(CX) & & & -\rho & = 0 & \\ y \in \mathbb{R}^m, & X \succeq 0, & \tau \geq 0, & S \succeq 0, & \rho \geq 0. & & \end{array}\right\} \quad (2.20)$$

A feasible solution to this system with $\tau > 0$ yields feasible solutions $\frac{1}{\tau}X$ and $\frac{1}{\tau}S$ to (P) and (D) respectively (by dividing the first two equations by τ). The last equation guarantees optimality by requiring a nonpositive duality gap. For this reason there is no interior solution to (2.20). The formulation (2.20) was first solved by Potra and Sheng [89] using an infeasible interior point method.

The *extended* self-dual embedding is treated here which has a known starting point on the central path. The advantage is that any feasible start path-following algorithm can be applied to such a problem. This is an important consideration in SDP, where many possible search directions and algorithms are available, with no clear method of choice at this time.

The strictly feasible, self–dual embedding is obtained by extending the constraint set (2.20) and adding extra variables to obtain:

$$\min_{y,X,\tau,\vartheta,S,\rho,\nu} \vartheta\beta$$

subject to

$$\left.\begin{array}{lllllll} & \mathbf{Tr}\,(A_iX) & -\tau b_i & +\vartheta\bar{b}_i & & & = 0 \quad \forall i \\ -\sum_{i=1}^m y_iA_i & & +\tau C & -\vartheta\bar{C} & -S & & = 0 \\ b^Ty & -\mathbf{Tr}\,(CX) & & +\vartheta\alpha & & -\rho & = 0 \\ -\bar{b}^Ty & +\mathbf{Tr}\,(\bar{C}X) & -\tau\alpha & & & -\nu & = -\beta \\ y\in\mathbb{R}^m, & X\succeq 0, & \tau\geq 0, & \vartheta\geq 0, & S\succeq 0, & \rho\geq 0, & \nu\geq 0 \end{array}\right\} \tag{2.21}$$

where

$$\begin{aligned} \bar{b}_i &:= b_i - \mathbf{Tr}\,(A_i), \quad i=1,\ldots,m \\ \bar{C} &:= C - I \\ \alpha &:= 1 + \mathbf{Tr}\,(C) \\ \beta &:= n+2. \end{aligned}$$

It is straightforward to verify that a feasible interior starting solution is given by $y^0 = 0$, $X^0 = S^0 = I$, and $\vartheta^0 = \rho^0 = \tau^0 = \nu^0 = 1$.

Self–duality of the embedding problem will now be proved. This means that both the embedding problem and (trivially) its dual satisfy the Slater condition. This is a sufficient condition for the existence of a complementary solution pair.

Theorem 2.4.1 *The embedding problem (2.21) is self–dual.*

Proof:

It will be shown that the Lagrangian dual of problem (2.21) is equivalent to problem (2.21). The Lagrangian of problem (2.21) is given by

$$\begin{aligned} L\,(\xi,\Omega,\kappa,\gamma,y,X,\tau,\vartheta,S,\rho,\nu) &= \vartheta\beta - \sum_{i=1}^m \xi_i\left(\mathbf{Tr}\,(A_iX) - \tau b_i + \vartheta\bar{b}_i\right) \\ &\quad +\mathbf{Tr}\,\left(\Omega\left(\sum_{i=1}^m y_iA_i - \tau C + \vartheta\bar{C} + S\right)\right) \end{aligned}$$

$$
\begin{aligned}
&-\kappa\left(b^T y - \mathbf{Tr}\left(CX\right) + \vartheta\alpha - \rho\right)\\
&-\gamma\left(-\bar{b}^T y + \mathbf{Tr}\left(\bar{C}X\right) - \tau\alpha - \nu + \beta\right),
\end{aligned}
$$

where the Lagrange multipliers $\xi \in \mathbb{R}^m$, $\Omega \in \mathcal{S}_n$, $\kappa \in \mathbb{R}$, and $\gamma \in \mathbb{R}$ have been introduced. The dual problem now becomes

$$
\max_{(\xi,\Omega,\kappa,\gamma)} \min_{(y,X,\tau,\vartheta,S,\rho,\nu)} L\left(\xi,\Omega,\kappa,\gamma,y,X,\tau,\vartheta,S,\rho,\nu\right) \tag{2.22}
$$

where $y \in \mathbb{R}^m$, $X \succeq 0$, $\tau \geq 0$, $\rho \geq 0$, $\nu \geq 0$, and $S \succeq 0$.

One can regroup the variables in the Lagrangian to obtain

$$
\begin{aligned}
L\left(\xi,\Omega,\kappa,\gamma,y,X,\tau,\vartheta,S,\rho,\nu\right) \;=\; & \sum_{i=1}^{m} y_i\left(\mathbf{Tr}\left(A_i\Omega\right) - b_i\kappa + \gamma\bar{b}_i\right)\\
& + \mathbf{Tr}\left(X\left[-\sum_{i=1}^{m}\xi_i A_i + \kappa C - \gamma\bar{C}\right]\right) + \rho\kappa\\
& + \tau\left(\sum_{i=1}^{m}\xi b_i - \mathbf{Tr}\left(C\Omega\right) + \gamma\alpha\right) + \nu\gamma\\
& + \vartheta\left(-\sum_{i=1}^{m}\xi\bar{b}_i - \mathbf{Tr}\left(\bar{C}\Omega\right) - \kappa\alpha + \beta\right) + \mathbf{Tr}\;\left(S\Omega\right) - \beta\gamma.
\end{aligned}
$$

This reformulation makes it clear that the inner minimization problem in (2.22) is only bounded from below if the following conditions hold:

$$
\left.
\begin{aligned}
\mathbf{Tr}\left(A_i\Omega\right) - b_i\kappa + \gamma\bar{b}_i &= 0, \quad i = 1,\ldots,m\\
-\textstyle\sum_{i=1}^{m}\xi_i A_i + \kappa C - \gamma\bar{C} &\succeq 0\\
\textstyle\sum_{i=1}^{m}\xi b_i - \mathbf{Tr}\left(C\Omega\right) + \gamma\alpha &\geq 0\\
-\textstyle\sum_{i=1}^{m}\xi\bar{b}_i - \mathbf{Tr}\left(\bar{C}\Omega\right) - \kappa\alpha + \beta &\geq 0\\
\kappa \geq 0,\ \gamma \geq 0,\ \Omega &\succeq 0.
\end{aligned}
\right\} \tag{2.23}
$$

Subject to these conditions, the inner minimization problem has minimum value

$$
\min_{(y,X,\tau,\vartheta,S,\rho,\nu)} L\left(\xi,\Omega,\kappa,\gamma,y,X,\tau,\vartheta,S,\rho,\nu\right) = -\beta\gamma,
$$

and the dual problem becomes

$$
\max -\beta\gamma
$$

subject to (2.23).

After changing to a minimization problem by inverting the sign of the objective, the dual becomes the embedding problem (2.21) where only the variables have been renamed and the slack variables have been omitted. □

The self duality implies that the duality gap is equal to $2\vartheta\beta$ and therefore $\vartheta^* = 0$ at an optimal solution. It is easy to show that

$$\vartheta\beta = \mathbf{Tr}\ (XS) + \tau\rho + \vartheta\nu. \tag{2.24}$$

This shows that an optimal solution satisfies the complementarity conditions:

$$\begin{aligned} XS &= 0 \\ \rho\tau &= 0 \\ \vartheta\nu &= 0. \end{aligned}$$

A maximally complementary solution of the embedding problem (2.21) can be used to obtain information about the original problem pair (P) and (D). In particular, one can distinguish between the three possibilities as discussed in the Introduction, namely

(I) A complementary solution pair (X^*, S^*) is obtained;

(II) A primal and/or dual improving ray is detected;

(III) A certificate is obtained that no complementary solution pair exists, and that neither (P) nor (D) has an improving ray.

Given a maximally complementary solution of the embedding problem, these cases are distinguished as follows.

Theorem 2.4.2 *Let $(y^*,\ X^*,\ \tau^*,\ \vartheta^*,\ S^*,\ \rho^*,\ \nu^*)$ be a maximally complementary solution to the self–dual embedding problem. Then:*

(i) if $\tau^* > 0$ then case (I) *holds;*

(ii) if $\tau^* = 0$ and $\rho^* > 0$ then case (II) *holds;*

(iii) if $\tau^* = \rho^* = 0$ *then case* (III) *holds.*

Proof:

Consider the two possibilities $\tau^* = 0$ and $\tau^* > 0$.

If $\tau^* > 0$, then $\frac{1}{\tau^*}X^*$ and $\frac{1}{\tau^*}S^*$ are maximally complementary and optimal for (P) and (D) respectively, *i.e.* case (I) holds.

If $\tau^* = 0$ then $\tau = 0$ in any optimal solution of the embedding problem. This implies that no pair of optimal solutions for (P) and (D) with duality gap zero exists, because if such a pair exists we can construct an optimal solution of the embedding problem with $\tau = 1$. If $\tau^* = 0$ it also follows that $\mathbf{Tr}\,(A_i X^*) = 0$ for all i and $\sum_{i=1}^m y_i^* A_i \preceq 0$. Now distinguish between two sub-cases: $\rho^* > 0$ and $\rho^* = 0$.

If $\rho^* > 0$ then $b^T y^* - \mathbf{Tr}\,(CX^*) > 0$, *i.e.* $b^T y^* > 0$ and/or $\mathbf{Tr}\,(CX^*) < 0$. In other words, there are primal and/or dual improving rays and case (II) applies. If $b^T y^* > 0$ then y^* is a dual improving ray. In this case (P) is infeasible, and if (D) is feasible it is unbounded. If $\mathbf{Tr}\,(CX^*) < 0$ then there exists a primal improving ray. In this case (D) is infeasible, and if (P) is feasible it is unbounded. If both $b^T y^* > 0$ and $\mathbf{Tr}\,(CX^*) < 0$ then both a primal and a dual improving ray exist and in this case both (P) and (D) are infeasible.

Conversely, one must show that if there exists a primal and/or dual improving ray, then any maximally complementary solution of the embedding problem must have $\rho^* > 0$ and $\tau^* = 0$. Given a primal improving ray $\bar{X} \succeq 0$, one can construct an optimal solution to the embedding by setting $X^* = \kappa \bar{X}$, where $\kappa > 0$ is a constant to be specified later, and further setting $\tau^* = 0$, $\vartheta^* = 0$ (which guarantees optimality), and $y^* = 0$, to obtain:

$$\begin{aligned} \rho^* &= -\kappa \mathbf{Tr}\,(C\bar{X}) > 0 \\ \kappa \mathbf{Tr}\,(A_i \bar{X}) = \mathbf{Tr}\,(A_i X^*) &= 0, \quad i = 1, \ldots, m \\ S^* &= 0 \\ \nu^* &= n + 2 + \kappa \mathbf{Tr}\left(C\bar{X} - \bar{X}\right). \end{aligned}$$

The first three equations show that ρ^*, X^* and S^* are feasible. It remains to prove that ν^* is nonnegative. This is ensured by choosing

$$\kappa = \frac{-1}{\mathbf{Tr}\left(C\bar{X} - \bar{X}\right)} > 0,$$

where the inequality follows from the definition of an improving ray. The proof for a dual improving ray proceeds analogously.

Finally, if a maximally complementary solution is obtained with $\tau^* = \rho^* = 0$, then we again have that all optimal solutions yield $\rho = \tau = 0$, *i.e.* cases (I) and (II) cannot occur. This completes the proof. □

Three important questions now arise:

- How is the embedding problem actually solved?
- How does one decide if $\tau^* > 0$ and $\rho^* > 0$ in a maximally complementary solution, if only an ϵ-optimal solution of the embedding problem is available?
- What additional information can be obtained if case (III) holds?

These three questions will be addressed in turn in the following three sections.

2.5 Solving the embedding problem

The embedding problem can be solved by any of the path following methods described in the next four chapters. To this end, one can relax the complementarity conditions of the embedding problem to

$$\begin{aligned} XS &= \mu I \\ \tau\rho &= \mu \\ \nu\vartheta &= \mu. \end{aligned}$$

If one defines new 'primal and dual variables' $\tilde{X}, \tilde{S}$ as follows:

$$\tilde{X} = \begin{bmatrix} X & & \\ & \tau & \\ & & \vartheta \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} S & & \\ & \rho & \\ & & \nu \end{bmatrix}, \tag{2.25}$$

then the centrality condition takes the familiar form: $\tilde{X}\tilde{S} = \mu I$. It follows from (2.24) that $\vartheta\beta = (n+2)\mu$ along the central path. This observation will be important in Section 2.7.

Furthermore, it is straightforward to verify that $\mathbf{Tr}\left(\Delta\tilde{X}\Delta\tilde{S}\right) = 0$, *i.e.* the orthogonality principle holds for the new variables. These two observations make the application of primal-dual path following methods straightforward: the search direction at a given point $(\tilde{X}, \tilde{S})$ can be computed from

$$\begin{array}{cccccccc}
 & \mathbf{Tr}\left(A_i\Delta X\right) & -\Delta\tau b_i & +\Delta\vartheta\bar{b}_i & & & = 0 & \forall i \\
-\sum_{i=1}^m \Delta y_i A_i & & +\Delta\tau C & -\Delta\vartheta\bar{C} & -\Delta S & & = 0 & \\
b^T\Delta y & -\mathbf{Tr}\left(C\Delta X\right) & & +\Delta\vartheta\alpha & & -\Delta\rho & = 0 & \\
-\bar{b}^T\Delta y & +\mathbf{Tr}\left(\bar{C}\Delta X\right) & -\Delta\tau\alpha & & & & -\Delta\nu = 0 &
\end{array}$$

and

$$\begin{aligned}
H_P\left(\Delta X S + X\Delta S\right) &= \mu I - H_P(XS) \\
\rho\Delta\tau + \tau\Delta\rho &= 2\mu - \tau\rho \\
\nu\Delta\vartheta + \vartheta\Delta\nu &= 2\mu - \vartheta\nu,
\end{aligned}$$

where H_P is the linear transformation given by

$$H_P(M) := \frac{1}{2}\left[PMP^{-1} + P^{-T}M^TP^T\right],$$

for any symmetric matrix M, and where the matrix P determines which symmetrization of the centrality condition is used. The best known choices for P were listed in Table 1.1 in Chapter 1. The proof of the existence and uniqueness of each of these search directions for the embedding problem requires some more mathematical formalism, and is given in Appendix C. The proof was first done in a general setting by Shidah *et al.* in [101], and for problems in standard form in [108].

Using these directions in conjunction with path–following methods, the embedding problem can be solved to ϵ-optimality with $O\left(\sqrt{n+2}\log(1/\epsilon)\right)$ worst-case iteration complexity. Suitable algorithms will be discussed in Chapters 3, 4, 5, and 6.

Remark: Note that ρ and τ can be viewed as eigenvalues of $\tilde{X}$ and $\tilde{S}$ respectively, corresponding to a fixed, shared eigenvector (see (2.25). This interpretation will be important in the next section.

2.6 Separating small and large variables

A path following interior point method only yields an ϵ-optimal solution to the embedding problem. This solution may yield small values of ρ and τ, and to distinguish between cases (I) to (III) it is necessary to know if these values are zero in a maximally complementary solution. This is the most problematic aspect of the analysis at this time, and only partial solutions are given here. Two open problems are stated which would help resolve the current difficulties.

In what follows the set of feasible $\tilde{X}$ for the embedding problems is denoted by $\tilde{\mathcal{P}}$ and the optimal set by $\tilde{\mathcal{P}}^*$. The sets $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^*$ are defined similarly. Finally, the size of the variables $\tilde{X}$ and $\tilde{S}$ is $\tilde{n} := n + 2$.

To separate 'small' and 'large' variables we need the following definition:

Definition 2.6.1 *The primal and dual condition numbers of the embedding are defined as*

$$\sigma_P := \max_{\tilde{X}\in\tilde{\mathcal{P}}^*} \min_{i:\lambda_i(\tilde{X})>0} \lambda_i(\tilde{X}), \quad \sigma_D := \max_{\tilde{S}\in\tilde{\mathcal{D}}^*} \min_{i:\lambda_i(\tilde{S})>0} \lambda_i(\tilde{S}),$$

The condition number σ of the embedding is defined as the minimum of these numbers $\sigma := \min\{\sigma_P, \sigma_D\}$.

Note that σ is well defined and positive because the solution set of the strictly feasible, self-dual embedding problem is compact (see *e.g.* [37]).

In linear programming a positive lower bound for σ can be given in terms of the problem data [98]. It is an open problem to give a similar bound in the semidefinite case:

Open problem 2.6.1 *Given strictly feasible SDP problems (P) and (D) one can define σ similarly to Definition 2.6.1. Using the notation of Definition 2.3.1, one can alternatively write*

$$\sigma = \max_{U_X, U_S, y} t$$

subject to

$$\begin{aligned} \mathbf{Tr}\left(A_i Q_B^* U_X {Q_B^*}^T\right) &= b_i, \quad i = 1, \ldots, m \\ \sum_{i=1}^m y_i A_i + Q_N^* U_S {Q_N^*}^T &= C \\ U_X \succeq tI, \qquad & U_S \succeq tI. \end{aligned}$$

Derive a lower bound for σ in terms of the problem data.

Other concepts of condition numbers for SDP are discussed in the paper by Freund [33].

If we have a centered solution to the embedding problem with centering parameter μ then we can use any knowledge of σ to decide the following:

Lemma 2.6.1 *For any positive μ one has:*

$$\tau(\mu) \geq \frac{\sigma}{\tilde{n}} \text{ and } \rho(\mu) \leq \frac{\tilde{n}\mu}{\sigma} \qquad \text{if } \tau^* > 0 \text{ and } \rho^* = 0$$
$$\tau(\mu) \leq \frac{\tilde{n}\mu}{\sigma} \text{ and } \rho(\mu) \geq \frac{\sigma}{\tilde{n}} \qquad \text{if } \tau^* = 0 \text{ and } \rho^* > 0,$$

where the superscript $*$ indicates a maximally complementary solution.

Proof:

Assume that ρ^* is positive in a maximally complementary solution. Let $\tilde{S}^* \in \tilde{\mathcal{D}}^*$ be such that ρ^* is as large as possible. By definition one therefore has $\rho^* \geq \sigma$. Recall form (2.9) that

$$\mathbf{Tr}\, \left(\tilde{X}(\mu)\tilde{S}^*\right) \leq \tilde{n}\mu,$$

which implies that the eigenvalues of $\tilde{X}(\mu)\tilde{S}^*$ satisfy

$$\lambda_i \left(\tilde{X}(\mu)\tilde{S}^*\right) \leq \tilde{n}\mu, \quad \forall\, i.$$

In particular

$$\tau(\mu)\rho^* \leq \tilde{n}\mu.$$

This shows that

$$\tau(\mu) \leq \frac{\tilde{n}\mu}{\rho^*} \leq \frac{\tilde{n}\mu}{\sigma}.$$

Since $\tau(\mu)\rho(\mu) = \mu$ one also has

$$\rho(\mu) \geq \frac{\sigma}{\tilde{n}}.$$

The case where $\tau^* > 0$ and $\rho^* = 0$ is proved in the same way. □

The lemma shows that once the barrier parameter μ has been reduced to the point where $\mu \leq \left(\frac{\sigma}{\tilde{n}}\right)^2$, then it is known which of τ or ρ is positive in a maximally complementary solution, provided that one is indeed positive. The case $\rho^* = \tau^* = 0$ cannot be detected using Lemma 2.6.1. It is an open problem to establish the convergence rate of τ and ρ in this case.

Open problem 2.6.2 *Consider the sequence $(X(\mu_t), S(\mu_t))$ for strictly feasible problems (P) and (D) as before, where*

$$\lambda_i(\mu_t)\sigma_i(\mu_t) = \mu_t, \quad i = 1, \ldots, n.$$

Recall that $T \subset \{1, \ldots, n\}$ denotes the index set where

$$\lambda_i(\mu_t) \to 0 \text{ and } \sigma_i(\mu_t) \to 0, \text{ as } t \to \infty \; \forall i \in T.$$

Establish an upper bound for $\lambda_i(\mu_t)$ and $\sigma_i(\mu_t)$ for $i \in T$ in terms of μ_t.

In a recent paper, Stoer and Wechs [103] consider the analogous problem in the case of *horizontal sufficient linear complementarity problems*, and prove a bound of $O(\sqrt{\mu})$.

The proof of Lemma 2.6.1 can easily be extended to the case where the ϵ-optimal solution is only approximately centered, where approximate centrality is defined by

$$\kappa(\tilde{X}, \tilde{S}) := \frac{\lambda_{\max}(\tilde{X}\tilde{S})}{\lambda_{\min}(\tilde{X}\tilde{S})} \leq \bar{\kappa},$$

for some parameter $\bar{\kappa} > 1$. Formally one has the following result.

Lemma 2.6.2 *Let $(\tilde{X}, \tilde{S})$ be a feasible solution of the embedding problem such that $\kappa(\tilde{X}, \tilde{S}) \leq \bar{\kappa}$ for some $\bar{\kappa} > 1$. One has the relations:*

$$\tau \geq \frac{\sigma}{\bar{\kappa}\tilde{n}} \text{ and } \rho \leq \frac{\mathbf{Tr}\,(\tilde{X}\tilde{S})}{\sigma} \qquad \text{if } \tau^* > 0 \text{ and } \rho^* = 0$$

$$\tau \leq \frac{\mathbf{Tr}\,(\tilde{X}\tilde{S})}{\sigma} \text{ and } \rho \geq \frac{\sigma}{\bar{\kappa}\tilde{n}} \qquad \text{if } \tau^* = 0 \text{ and } \rho^* > 0$$

where the superscript $*$ indicates a maximally complementary solution.

2.7 Remaining duality and feasibility issues

If $\rho^* = \tau^* = 0$ in a maximally complementary solution of the embedding problem (*i.e.* case (III) holds), then one of the following situations has occurred:

1) The problems (P) and (D) are solvable but have a positive duality gap;

2) either (P) or (D) (or both) are weakly infeasible;

3) both (P) and (D) are feasible, but one or both are unsolvable.

Case 2) was illustrated in Example 2.2.1. The remaining two cases occur in the following examples:

Example 2.7.1 *The following problem (adapted from [112]) which is in the form (D): find*

$$\sup y_2$$

subject to

$$y_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is solvable with optimal value $y_2^* = 0$ but the corresponding primal problem has optimal value 1. □

Example 2.7.2 *Another difficulty is illustrated by the following problem (adapted from [112]): find*

$$\sup y_2$$

subject to

$$y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \preceq \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

which is not solvable but $\sup_{y \in \mathcal{D}} y_2 = 1$. The corresponding primal problem is solvable with optimal value 1. □

The aim is therefore to see what further information can be obtained in the case $\tau^* = \rho^* = 0$. To this end, recall that along the central path of the embedding problem one has

$$\rho(\mu_t)\tau(\mu_t) = \mu_t \text{ and } \vartheta(\mu_t)\beta = \tilde{n}\mu_t \tag{2.26}$$

which shows that $\rho(\mu_t) \to \rho^* = 0$ implies

$$\vartheta(\mu_t)/\tau(\mu_t) \to 0 \text{ as } t \to \infty. \tag{2.27}$$

This shows (by (2.21)) that:

$$\mathbf{Tr}\left(\frac{1}{\tau(\mu_t)}A_i X(\mu_t)\right) \to b_i, \quad \forall i \tag{2.28}$$

and

$$\sum_{i=1}^{m} \frac{y_i(\mu_t)}{\tau(\mu_t)}A_i + \frac{1}{\tau(\mu_t)}S(\mu_t) \to C. \tag{2.29}$$

In other words if either or both of the sequences

$$\left\{\frac{1}{\tau(\mu_t)}X(\mu_t)\right\} \text{ and } \left\{\frac{1}{\tau(\mu_t)}S(\mu_t)\right\} \tag{2.30}$$

converge, the limit is feasible for (P) or (D) respectively. On the other hand, if (2.28) (resp. (2.29)) holds but (P) (resp. (D)) is infeasible, then (P) (resp. (D)) is weakly infeasible. If one also has

$$\frac{\rho(\mu_t)}{\tau(\mu_t)} \to 0 \text{ as } t \to \infty \tag{2.31}$$

then it also follows from (2.21) that

$$\frac{1}{\tau(\mu_t)}b^T y(\mu_t) - \frac{1}{\tau(\mu_t)}\mathbf{Tr}\,(CX(\mu_t)) \to 0.$$

If this happens, at least one of the sequences in (2.30) diverges (or else an optimal pair with zero duality gap exists).

On the other hand, one always has $\vartheta(\mu_t)/\rho(\mu_t) \to 0$ if $\tau(\mu_t) \to 0$, from (2.26). If it also holds that

$$\frac{\tau(\mu_t)}{\rho(\mu_t)} \to 0 \text{ as } t \to \infty \tag{2.32}$$

then

$$\frac{1}{\rho(\mu_t)} b^T y(\mu_t) - \frac{1}{\rho(\mu_t)} \mathbf{Tr}\ (CX(\mu_t)) \to 1,$$

$$\mathbf{Tr}\left(\frac{1}{\rho(\mu_t)} A_i X(\mu_t)\right) \to 0, \quad \forall i \tag{2.33}$$

and

$$\sum_{i=1}^{m} \frac{y_i(\mu_t)}{\rho(\mu_t)} A_i + \frac{1}{\rho(\mu_t)} S(\mu_t) \to 0. \tag{2.34}$$

A so-called *asymptotic improving ray* (or weak improving ray) is thus detected for (P) and/or (D). It is easy to show (see *e.g.* [69]) that an asymptotic improving ray in (P) (resp. (D)) implies weak infeasibility in (D) resp. (P).

The problem is that none of these indicators gives a certificate of the status of a given problem. For example, there is no guarantee that (2.32) will hold if one (or both) of (P) and (D) have weak improving rays. Luo *et al.* [69] derive similar detectors and show that these detectors yield no information in some cases. We therefore need to go a step further, by replacing the embedding of (P) and (D) with a different embedding problem where 'stronger' duals are embedded. This is the subject of the next section.

2.8 Embedding extended Lagrange-Slater duals

Assume now that the aim is to solve a given problem (D) in the standard dual form, like the problems in the examples.[5] In other words, the goal is to find the value

$$d^* = \sup_{S,y \in \mathcal{D}} b^T y$$

if it is finite or obtain certificate that (D) is infeasible, or alternatively, a certificate of unboundedness.

For the example problems the embedding of (D) and its Lagrangian dual (P) will be insufficient for this purpose. The solution proposed here is to solve a second embedding problem, using so-called *extended Lagrange-Slater duals*. To this end, the so-called gap-free primal problem (P_{gf}) of (D) may be formulated instead of

[5] The case where the problem under consideration is in the standard primal form can be treated analogously. This is described in Section 2.10.1.

using the standard primal problem (P). The gap-free primal was first formulated by Ramana [92], and takes the form:

$$\min \mathbf{Tr}\ (C(U_0 + W_k))$$

subject to

$$\begin{array}{rcll} \mathbf{Tr}\ (A_j(U_0 + W_k)) & = & b_i, & j = 1, \ldots, m \\ \mathbf{Tr}\ (C(U_i + W_{i-1})) & = & 0, & i = 1, \ldots, k \\ \mathbf{Tr}\ (A_j(U_i + W_{i-1})) & = & 0, & i = 1, \ldots, k, \quad j = 1, \ldots, m \\ W_0 & = & 0, & \\ \begin{bmatrix} I & W_i^T \\ W_i & U_i \end{bmatrix} & \succeq & 0, & i = 1, \ldots, k, \\ U_0 & \succeq & 0, & \end{array}$$

where the variables are $U_i \succeq 0$, and $W_i \in \mathbb{R}^{n\times n},\ i = 0, \ldots, k$. The parameter k determines the size of this problem and satisfies:

$$k \leq \min\{m, n\}.$$

It is determined by the *degree of regularity* of our problem (D), *i.e.* the dimension of the minimal face[6] (say $\mathcal{F}_D$) of the cone $\mathcal{S}_n^+$ which contains the feasible set of (D). The constraint set of (P_{gf}) implicitly defines $\mathcal{F}_D$ [94]. Note that if $k = 0$ the standard Lagrangian dual (P) of (D) is obtained; this corresponds to the case where $\mathcal{F}_D = \mathcal{S}_n^+$, *i.e.* where (D) satisfies the Slater condition.

Note that the gap-free primal problem is easily cast in the standard primal form. Moreover, its size is polynomial in the size of (D). Unlike the standard primal (P), (P_{gf}) has the following desirable features:

- (Weak duality) If $(y, S) \in \mathcal{D}$ and $(U_i, W_i),\ i = 0, \ldots, m$ is feasible for (P_{gf}) then $b^T y \leq \mathbf{Tr}\ (C(U_0 + W_m))$.
- (Dual boundedness) If (D) is feasible, its optimal value is finite if and only if (P_{gf}) is feasible.

[6] We say that a set $\mathcal{F} \subset \mathcal{S}_n^+$ is a face of $\mathcal{S}_n^+$ if $X_1, X_2 \succeq 0$ and $X_1 + X_2 \in \mathcal{F}$ implies $X_1, X_2 \in \mathcal{F}$.

- (Zero duality gap) The supremal value of (P_{gf}) equals the infimum value of (D) if and only if both (P_{gf}) and (D) are feasible.
- (Attainment) If the supremum value of (D) is finite, then it is attained by (P_{gf}).

The standard (Lagrangian) dual problem associated with (P_{gf}) is called the *corrected dual* (D_{cor}). The surprising result is that the pair (P_{gf}) and (D_{cor}) are now 'gap-free' [93], *i.e.* (2.1) is satisfied.

Moreover, a feasible solution to (D) can be extracted from a feasible solution to (D_{cor}). The only problem is that (D_{cor}) does not necessarily attain its supremum, even if (D) does.

A natural question is whether (D_{cor}) is strongly infeasible if (D) is only weakly infeasible. This would simplify matters greatly as strong infeasibility can be detected more easily. Unfortunately this is not the case, as the following example shows.

Example 2.8.1 *The gap-free dual of the weakly infeasible problem* (D) *in Example 2.2.1 is given by: minimize*

$$\mathbf{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(U_0+W)\right) = 2([U_0]_{12}+[W]_{12})$$

subject to

$$\begin{aligned}
\mathbf{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}(U_0+W)\right) &= [U_0]_{11}+[W]_{11}=1,\\
\mathbf{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}U_1\right) &= [U_1]_{11}=0,\\
\mathbf{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}U_1\right) &= 2[U_1]_{12}=0,\\
\begin{bmatrix} I & W^T \\ W & U_1 \end{bmatrix} &\succeq 0,\\
W\in\mathbb{R}^{2\times 2},\ U_0\succeq 0,\ U_1\succeq 0.&
\end{aligned}$$

Note that the variables in U_1 do not appear in the objective. Since the only nonzero element of U_1 is $[U_1]_{22}$, it follows from the fourth constraint that the only nonzero entry in W is $[W]_{22}$. This element also does not appear in the objective. One can therefore eliminate U_1 and W to obtain the problem

$$\min \mathbf{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U_0\right)$$

subject to

$$\mathbf{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_0\right) = 1, \quad U_0 \succeq 0,$$

which is simply the Lagrangian dual of (D). □

In what follows the embedding problem is solved using (P_{gf}) and (D_{cor}) for the problem (D). It is therefore assumed that the solution of the embedding of (D) and its Lagrangian dual (P) has yielded $\tau^* = \rho^* = 0$. It is therefore already known that (D) is not strongly infeasible.

Three possibilities remain:

(i) the problem (D) is feasible and has a finite optimal value;

(ii) the problem (D) is feasible and unbounded but does not have an improving ray;

(iii) the problem (D) is weakly infeasible.

If (and only if) case (i) holds, then (P_{gf}) and (D_{cor}) will have the same (finite) optimal values (zero duality gap). Problem (P_{gf}) will certainly attain this common optimal value, but (D_{cor}) may not. The possible duality relations are listed in Table 2.1.

In what follows the variables $(y,\ X,\ \tau,\ \vartheta,\ S,\ \rho,\ \nu)$ refer to the embedding of (P_{gf}) and (D_{cor}). The feasible sets of (P_{gf}) and (D_{cor}) are denoted by $\mathcal{P}_{gf}$ and $\mathcal{D}_{cor}$ respectively. The subscripts 'gf' and 'cor' refer to the variables of (P_{gf}) and (D_{cor}) respectively, but the problem data for (P_{gf}) and (D_{cor}) will still be denoted by C, b, A_i for simplicity.

Status of (D)	Status of (P_{gf})	Status of (D_{cor})
$d^* < \infty$	$p^*_{gf} = d^*$	$d^*_{cor} = d^*$
unbounded	infeasible	unbounded
infeasible	unbounded	infeasible

Table 2.1: Duality relations for a given problem (D), its gap-free dual (P_{gf}) and its corrected problem (D_{cor}).

The aim is to identify or exclude the general situation where (P_{gf}) and (D_{cor}) are such that

$$\sup_{y_{cor}, S_{cor} \in \mathcal{D}_{cor}} b^T y_{cor} = \min_{X_{gf} \in \mathcal{P}_{gf}} \mathbf{Tr}\,(C X_{gf}), \tag{2.35}$$

and the optimal value $\sup_{y_{cor}, S_{cor} \in \mathcal{D}_{cor}} b^T y_{cor}$ may or may not be attained.

If the optimal value of (D_{cor}) is attained, the embedding yields a solution with $\tau^* > 0$ and we are done. Similarly, if $\tau^* = 0$ and $\rho^* > 0$, an improving ray is detected and the status of (D) follows from Table 2.1. It is therefore sufficient to consider the case where the embedding of (P_{gf}) and (D_{cor}) has $\tau^* = \rho^* = 0$ in a maximally complementary solution.

To this end, it is first shown that (2.31) must hold if d^* is finite.

Lemma 2.8.1 *Assume that a given problem (D) has finite optimal value d^*. Then (2.31) holds for the embedding of (P_{gf}) and (D_{cor}).*

Proof:

Let $\epsilon_t := \vartheta(\mu_t)/\tau(\mu_t)$ and $(X_{gf}, y_{cor}, S_{cor}) \in \mathcal{P}_{gf} \times \mathcal{D}_{cor}$. Note that $\epsilon_t \to 0$ as $t \to \infty$ by (2.27). The following notation is introduced to simplify the expressions:

$$X_t := \frac{1}{\tau(\mu_t)} X(\mu_t), \quad S_t := \frac{1}{\tau(\mu_t)} S(\mu_t).$$

In terms of this notation one has from (2.21):

$$\begin{aligned} \mathbf{Tr}\ (A_i X_t) + \epsilon_t \bar{b}_i &= b_i, \\ \sum_{i=1}^{m} (y_t)_i A_i + S_t + \epsilon_t \bar{C} &= C. \end{aligned}$$

Using the feasibility of X_{gf} and S_{cor} it is easy to show that

$$\mathbf{Tr}\ (X_t S_{cor} + S_t X_{gf}) = \mathbf{Tr}\,(X_{gf} S_{cor}) - \epsilon_t \mathbf{Tr}\,(\bar{C} X_{gf}) + \epsilon_t \bar{b}^T y_{cor} + \Big[\mathbf{Tr}\,(C X_t) - b^T y_t\Big].$$

Substitution of $\bar{b}_i = b_i - \mathbf{Tr}\,(A_i)$ and $\bar{C} = C - I$, and using

$$\mathbf{Tr}\,(S_{cor}) = \mathbf{Tr}\,\left(C - \sum_{i=1}^{m} (y_{cor})_i A_i\right)$$

yields

$$\mathbf{Tr}\ (X_t S_{cor} + S_t X_{gf}) = (1+\epsilon_t)\mathbf{Tr}\,(X_{gf} S_{cor}) - \epsilon_t \mathbf{Tr}\,(X_{gf} + S_{cor}) - \epsilon_t \mathbf{Tr}\,(C) + \mathbf{Tr}\,(C X_t) - b^T y_t. \tag{2.36}$$

If (2.31) does not hold, then there exists an $\bar{\epsilon} > 0$ such that

$$\mathbf{Tr}\,(C X_{\bar{t}}) - b^T y_{\bar{t}} < -\bar{\epsilon} \tag{2.37}$$

for some $\bar{t}$ which can be chosen arbitrarily large.

Since X_{gf} and S_{cor} were arbitrary one can assume that $\mathbf{Tr}\,(X_{gf} S_{cor}) < \bar{\epsilon}/2$. Choose $\bar{t}$ such that (2.37) holds and

$$\epsilon_{\bar{t}} \mathbf{Tr}\,(X_{gf} S_{cor}) - \epsilon_{\bar{t}} \mathbf{Tr}\,(X_{gf} + S_{cor}) - \epsilon_{\bar{t}} \mathbf{Tr}\,(C) < \bar{\epsilon}/2.$$

The left hand side of (2.36) is always nonnegative, while the right hand side is negative for the above choice of $\bar{t}$. This contradiction shows that if a pair (X_{gf}, S_{cor}) exists with arbitrarily small duality gap, then (2.31) must hold.

This completes the proof, since (P_{gf}) and (D_{cor}) are feasible with zero gap if and only if (D) is feasible with finite optimal value. □

The next question is how to obtain the value d^* if it is finite. The following lemma shows that this value can be obtained from a sequence of centered iterates of the embedding as a limit value.

Lemma 2.8.2 *Assume the optimal value of* (D) *to be finite,* i.e. $d^* < \infty$*, and let* X^*_{gf} *be an optimal solution of* (P_{gf})*. One now has*

$$d^* = \mathbf{Tr}\,\left(C X^*_{gf}\right) = \lim_{t\to\infty} \frac{1}{\tau(\mu_t)} b^T y(\mu_t) = \lim_{t\to\infty} \mathbf{Tr}\,\left(\frac{C X(\mu_t)}{\tau(\mu_t)}\right).$$

Proof:

Let X^*_{gf} be any optimal solution of (P_{gf}). (Recall that (P_{gf}) is always solvable and its optimal value equals the optimal value of (D)). Using the 'subscript t' notation from the previous lemma, and the statement of the self–dual problem in (2.21), one can easily show that

$$\mathbf{Tr}\left(X^*_{gf}S_t\right) = \mathbf{Tr}\,(CX^*) - b^T y_t - \epsilon_t \mathbf{Tr}\left(\bar{C}X^*_{gf}\right)$$

or

$$\begin{aligned}\left|\mathbf{Tr}\left(CX^*_{gf}\right) - b^T y_t\right| &= \left|\mathbf{Tr}\left(X^*_{gf}S_t\right) - \epsilon_t \mathbf{Tr}\left(\bar{C}X^*_{gf}\right)\right| \\ &\leq \left|\mathbf{Tr}\left(X^*_{gf}S_t\right)\right| + \left|\epsilon_t \mathbf{Tr}\left(\bar{C}X^*_{gf}\right)\right|.\end{aligned}$$

The second right hand side term converges to zero as $\epsilon_t \to 0$. The first right hand side term can be made arbitrarily small, as can easily be seen from (2.36). This completes the proof. □

We can now show how to detect infeasibility or unboundedness of (D). Recall that if

$$\lim_{t\to\infty} \frac{\tau(\mu_t)}{\rho(\mu_t)} = 0 \tag{2.38}$$

then an asymptotic improving ray is detected for (D_{cor}) or (P_{gf}). This implies weak infeasibility of either (D_{cor}) or (P_{gf}), and thus the status of (D) is known from Table 2.1. The possible combinations are listed in Table 2.2. The status of

Status of (D)	$\lim_{\mu\to 0} \mathbf{Tr}\left(\frac{CX(\mu)}{\rho(\mu)}\right)$	$\lim_{\mu\to 0} \frac{b^T y(\mu)}{\rho(\mu)}$
unbounded	$[0,\infty)$	$(-\infty,0)$
infeasible	$(-\infty,0)$	$[0,\infty)$

Table 2.2: Indicators of the status of problem (D) via the embedding of (D_{cor}) and (P_{gf}), for the case where $\lim_{\mu\to 0} \frac{\tau(\mu)}{\rho(\mu)} = 0$. In this case d^* cannot be finite.

(D) is known even if (2.38) does not hold for the embedding of (P_{gf}) and (D_{cor}). This is proved in the following lemma.

Lemma 2.8.3 *Assume that (D) does not have a finite optimal value. The status of (D) is then decided as follows:*

$$\lim_{\mu\to 0}\mathbf{Tr}\left(\frac{CX(\mu)}{\tau(\mu)}\right)=\lim_{\mu\to 0}\mathbf{Tr}\left(\frac{b^Ty(\mu)}{\tau(\mu)}\right) \;=\; \infty \text{ if } (D) \text{ is unbounded;}$$
$$\lim_{\mu\to 0}\mathbf{Tr}\left(\frac{CX(\mu)}{\tau(\mu)}\right)=\lim_{\mu\to 0}\mathbf{Tr}\left(\frac{b^Ty(\mu)}{\tau(\mu)}\right) \;=\; -\infty \text{ if } (D) \text{ is infeasible.}$$

Proof:

We first consider the case where (D) is infeasible. Recall from Table 2.1 that (D) is infeasible if and only if (P_{gf}) is unbounded. Let us therefore assume that (P_{gf}) is unbounded, and let $K>0$ be given. By the assumption, there exists a $X_{gf}\in\mathcal{P}_{gf}$ such that $\mathbf{Tr}\,(CX_{gf})<-K$. It is straightforward to derive the following relation from the statement of the self–dual problem (2.21):

$$\begin{aligned}\frac{1}{\tau(\mu_t)}\mathbf{Tr}\,(CX(\mu_t)) &= \mathbf{Tr}\,(CX_{gf})+\frac{\vartheta(\mu_t)}{\tau(\mu_t)}\alpha-\frac{\rho(\mu_t)}{\tau(\mu_t)}-\frac{\vartheta(\mu_t)}{\tau(\mu_t)}\mathbf{Tr}\,(\bar{C}X_{gf})\\ &\quad -\frac{1}{\tau(\mu_t)}\mathbf{Tr}\,(S(\mu_t)X_{gf})\\ &\leq -K+\frac{\vartheta(\mu_t)}{\tau(\mu_t)}\alpha-\frac{\vartheta(\mu_t)}{\tau(\mu_t)}\mathbf{Tr}\,(\bar{C}X_{gf}),\end{aligned}$$

where two nonpositive terms were discarded to obtain the inequality. Since

$$\lim_{t\to\infty}\frac{\vartheta(\mu_t)}{\tau(\mu_t)}=0$$

we have for t 'large enough':

$$\frac{1}{\tau(\mu_t)}\mathbf{Tr}\,(CX(\mu_t))\leq -K.$$

Since $K>0$ was arbitrary, it follows that:

$$\lim_{\mu\to 0}\mathbf{Tr}\left(\frac{CX(\mu)}{\tau(\mu)}\right)=-\infty.$$

One also has

$$\lim_{\mu\to 0}\mathbf{Tr}\left(\frac{b^Ty(\mu)}{\tau(\mu)}\right)=-\infty,$$

by repeating the proof using

$$\frac{b^T y(\mu_t)}{\tau(\mu_t)} = \mathbf{Tr}\,(CX_{gf}) - \frac{1}{\tau(\mu_t)}\mathbf{Tr}\;(S(\mu_t)X_{gf}) - \frac{\vartheta(\mu_t)}{\tau(\mu_t)}\mathbf{Tr}\,(\bar{C}X_{gf}).$$

This completes the proof that:

$$\lim_{\mu\to 0}\mathbf{Tr}\;\left(\frac{CX(\mu)}{\tau(\mu)}\right) = \lim_{\mu\to 0}\mathbf{Tr}\;\left(\frac{b^T y(\mu)}{\tau(\mu)}\right) = -\infty,$$

if (D) is infeasible. The case where (D) is unbounded is proved in a similar way. □

In Table 2.3 the results of the lemma are summarized. The only question that

Status of (D)	$\lim_{\mu\to 0}\frac{\rho(\mu)}{\tau(\mu)}$	$\lim_{\mu\to 0}\mathbf{Tr}\;\left(\frac{CX(\mu)}{\tau(\mu)}\right)$	$\lim_{\mu\to 0}\frac{b^T y(\mu)}{\tau(\mu)}$
$d^* < \infty$	0	d^*	d^*
unbounded	$[0,\infty)$	∞	∞
infeasible	$[0,\infty)$	$-\infty$	$-\infty$

Table 2.3: Indicators of the status of problem (D) via the embedding of (D_{cor}) and (P_{gf}).

cannot be answered by this analysis is whether or not (D) actually attains its optimal value, if it is finite. This question can be answered by solving a third embedding problem, where (D) and (P_{gf}) are combined as a single SDP problem, with the zero objective function and the added constraint that the objective values of (D) and (P_{gf}) must be equal. The resulting SDP problem is feasible if and only if (D) attains its optimal value, and infeasibility can be detected as described in this chapter.

The need for three embedding problems (in the worst-case) is somewhat unfortunate, and unifying the approach such that one embedding is sufficient remains a topic for future research.

2.9 Summary and discussion

The embedding strategy yields an ϵ-optimal solution of a given semidefinite program and its Lagrangian dual in $O\left(\sqrt{n+2}\log(1/\epsilon)\right)$ iterations, provided a complementary solution pair exists. If no complementary pair exists, strong infeasibility is detected instead, if it occurs. The underlying assumption is that enough information concerning a maximally complementary solution of the embedding problem can be obtained from an ϵ-optimal solution. This issue is not yet satisfactorily resolved.

If neither strong infeasibility nor a complementary solution pair is found, a second embedding problem can be solved using extended Lagrange-Slater dual problems instead of standard (Lagrangian) duals. This embedding is used to generate sequences in terms of which weak infeasibility or a (finite) optimal value of a given problem can be characterized asymptotically. In this way certificates (of infinite length) of infeasibility and unboundedness can be detected or the optimal value can be obtained.

These results cast some light on the best-known problem in the complexity theory of SDP, namely the semidefinite feasibility problem (SDFP).

Definition 2.9.1 (Semidefinite feasibility problem) *Determine if the linear matrix inequality:*

$$F(y) := C - \sum_{i=1}^{m} y_i A_i \succeq 0,$$

with rational data allows a feasible solution.

It was shown by Ramana [92] that this problem is not NP-complete unless NP $=$ co-NP.

Note that the existence of a feasible solution to $F(y) \succeq 0$ is equivalent to the existence of a complementary solution pair to the problem

$$\max_{y} \left\{ 0 \ : \ C - \sum_{i=1}^{m} y_i A_i \succeq 0 \right\}$$

and its Lagrangean dual

$$\min \left\{ \mathbf{Tr}\left(CX\right) \ : \ \mathbf{Tr}\left(A_i X\right) = 0, \ X \succeq 0 \right\}.$$

A complementary solution pair can be detected (or the possibility excluded) in polynomial time by using the embedding approach, if the open problems 2.6.1 and 2.6.2 can be solved.

2.10 Extensions

In this section we show how to apply the embedding strategy if the problem to be solved is in one of the following three forms:

1. an SDP in the standard primal form (P) (instead of the form (D) which was considered in Section 2.8);
2. a SDP problem in the so-called *symmetric form*;
3. a general convex optimization problem in conic form (not necessarily SDP).

2.10.1 Solving problems in the standard primal form (P)

The analysis of Section 2.8 can also be performed for the case where the problem under consideration is of the standard primal form (P). The ELSD dual of (P) takes the form [94]:

$$\max b^T y^0$$

subject to

$$\begin{aligned}
\sum_{i=1}^{m} y_i^0 A_i + Z_k + Z_k^T &\preceq C \\
\sum_{i=1}^{m} y_i^j A_i + Z_{j-1} + Z_{j-1}^T &\succeq 0, \quad j = 1, \ldots, k \\
b^T y^j &= 0, \quad j = 1, \ldots, k \\
\begin{bmatrix} I & Z_j^T \\ Z_j & \sum_{i=1}^{m} y_i^j A_i \end{bmatrix} &\succeq 0, \quad j = 1, \ldots, k \\
Z_0 &= 0,
\end{aligned}$$

where the variables are $y^j \in \mathbb{R}^m$ and $Z_j \in \mathbb{R}^{n \times n}$, $j = 0, \ldots, m$, and $k \leq \min\{m, n\}$ is now determined by the degree of regularity of (P).

This 'gapfree dual' of (P), which we denote by (D_{gf}), may be cast in the standard dual form (D). It has the same features as listed for (P_{gf}) in Section 2.8. We can therefore embed (D_{gf}) and its Lagrangian dual (which is the 'corrected problem' of (P), say (P_{cor})) to obtain a certificate of the status of our problem (P).

2.10.2 The embedding for problems in symmetric form

Assume that the goal is to solve an SDP problem in the *symmetric form*:

$$(\hat{\text{P}}): \quad \min_X \{ \mathbf{Tr}\,(CX) \ : \ \mathbf{Tr}\,(A_i X) \geq b_i, \ i = 1, \ldots, m, \ X \succeq 0 \}\,,$$

which has the associated Lagrangian dual problem

$$(\hat{\text{D}}): \quad \max_{y,S} \left\{ b^T y \ : \ \sum_{i=1}^{m} y_i A_i + S = C, \ S \succeq 0, \ y \geq 0 \right\}.$$

As for the *standard form* problems (P) and (D), the pair of problems $(\hat{\text{P}})$ and $(\hat{\text{D}})$ may be embedded in a self–dual SDP problem with known central starting point.[7] This self–dual embedding for $(\hat{\text{P}})$ and $(\hat{\text{D}})$ takes the form:

$$\min_{y,X,\tau,\vartheta,z,S,\rho,\nu} \vartheta\beta$$

subject to

$$\begin{array}{cccccccccc}
 & \mathbf{Tr}\,(A_i X) & -\tau b_i & +\vartheta \bar{b}_i & -z_i & & & & = 0 & \forall i \\
-\sum_{i=1}^m y_i A_i & & +\tau C & -\vartheta \bar{C} & & -S & & & = 0 & \\
b^T y & -\mathbf{Tr}\,(CX) & & +\vartheta\alpha & & & -\rho & & = 0 & \\
-\bar{b}^T y & +\mathbf{Tr}\,(\bar{C}X) & -\tau\alpha & & & & & -\nu & = -\beta & \\
y \geq 0, & X \succeq 0, & \tau \geq 0, & \vartheta \geq 0, & z \geq 0, & S \succeq 0, & \rho \geq 0, & \nu \geq 0 & &
\end{array}$$

where

$$\bar{b}_i \ := \ b_i + 1 - \mathbf{Tr}\, A_i$$

[7] The proofs are analogous to the proofs given in this chapter, and may be found in De Klerk *et al.* [22].

$$\begin{aligned} -\bar{C} &:= I + \sum_{i=1}^{m} A_i - C \\ \alpha &:= 1 + \mathbf{Tr}\, C - b^T e \\ \beta &:= 2n + 2. \end{aligned}$$

It is straightforward to verify that a feasible interior starting solution is given by the centered point $y^0 = z^0 = e$, $X^0 = S^0 = I$, $\vartheta^0 = \rho^0 = \tau^0 = \nu^0 = 1$, where $e \in \mathbb{R}^m$ denotes the all–one vector.

If one defines the new block diagonal matrix variable

$$\bar{X} = \begin{bmatrix} \operatorname{diag} y & & & & & & & \\ & X & & & & & & \\ & & \tau & & & & & \\ & & & \vartheta & & & & \\ & & & & \operatorname{diag} z & & & \\ & & & & & S & & \\ & & & & & & \rho & \\ & & & & & & & \nu \end{bmatrix} \succeq 0$$

then the self–dual problem may easily be cast in the standard primal form (P). The resulting problem has $2m + 2n^2 + 4$ variables and the same number of constraints. It may be solved with the primal algorithms which will be described in Chapter 5. Note that – given an original problem in symmetric form – the algorithm now functions as a primal-dual, infeasible-start algorithm, while the worst case complexity bound becomes $O(\sqrt{n + m + 2}L)$ iterations. This is the usual worst–case complexity bound for the solution of problems in the symmetric form (P̂),(D̂). (In order to transform a problem (P̂) in symmetric from to the standard form (P), m slack variables must be added to change the inequality constraints into equality constraints. The nonnegativity of the slack variables is then ensured by adding the slacks as extra diagonal elements of X. This increases the dimension of (P) from n to $n + m$.) The worst-case complexity of the primal–dual, infeasible–start algorithm obtained in this way is therefore the same as that of the infeasible start method of Potra and Sheng [89] when applied to problems in symmetric form.

Conversely, if one starts with a pair of problems in the standard form (P) and (D), some strategy is needed to convert it to the symmetric form in order to apply the embedding strategy. The trivial transformation

$$\mathbf{Tr}\,(A_iX) = b_i \Rightarrow \mathbf{Tr}\,(A_iX) \leq b_i,\ \mathbf{Tr}\,(A_iX) \geq b_i,\ i = 1, \ldots, m$$

only preserves the $O(\sqrt{n}L)$ iteration complexity bound if $m = O(n)$. In the worst case one can have $m = O(n^2)$ in which case the complexity becomes $O(nL)$. A transformation which reduces problem size is therefore needed if $m = O(n^2)$.

It is well known that such an efficient transformation from the standard to the symmetric form can be done in linear programming (see *e.g.* [98]), but this remains a topic for future research in the SDP case.

2.10.3 Embedding general conic convex problems

The results of the previous sections can be generalized to primal–dual convex problems in the conic formulation. Consider the primal problem as

$$\min_x \{\ c^T x\ :\ Ax - b \in \mathcal{C}_1,\ \ x \in \mathcal{C}_2\ \}$$

and its dual problem as

$$\max_y \{\ b^T y\ :\ -A^T y + c \in \mathcal{C}_2^*,\ \ y \in \mathcal{C}_1^*\ \}$$

where $\mathcal{C}_1,\ \mathcal{C}_2$ are convex cones, $\mathcal{C}_1^*,\ \mathcal{C}_2^*$ are their dual cones respectively, A is an $m \times n$ matrix, $b, y \in \mathbb{R}^m$ and $c, x \in \mathbb{R}^n$. These problems can be embedded in the skew–symmetric self–dual problem with nonempty interior as follows.

$$\min_{x,y,\tau,\vartheta} \vartheta\beta$$

subject to

$$\begin{array}{rrrrcr}
 & Ax & -\tau b & +\vartheta\bar{b} & \in & \mathcal{C}_1 \\
-A^T y & & +\tau c & -\vartheta\bar{c} & \in & \mathcal{C}_2^* \\
b^T y & -c^T x & & +\vartheta\alpha & \geq & 0 \\
-\bar{b}^T y & +\bar{c}^T x & -\tau\alpha & & \geq & -\beta \\
y \in \mathcal{C}_1^*, & x \in \mathcal{C}_2, & \tau \geq 0, & \vartheta \geq 0, & &
\end{array}$$

where

$$\begin{aligned} \bar{b} &:= b + e_m - Ae_n \\ -\bar{c} &:= e_n + A^T e_m - c \\ \alpha &:= 1 + c^T e_n - b^T e_m \\ \beta &:= m + n + 2, \end{aligned}$$

where e_m and e_n denote the all–one vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

One can show via Lagrangian duality that this embedding problem is self–dual with the all one solution as an initial interior feasible solution. Analogous results can be derived as for the positive semidefinite case — maximal complementarity of the scalar variables (which are in $\mathbb{R}_+$) can be proved the same way as in the SDP case (see De Klerk *et al.* [22]). This is sufficient to prove the validity of the above embedding. The general embedding was recently studied in more detail by Luo *et al.* [69].

Chapter 3

Primal–dual Affine–Scaling Methods

Two primal–dual affine–scaling algorithms for linear programming are extended to semidefinite programming. The first algorithm is the Dikin-type primal–dual affine–scaling method of Jansen et al. *[49] and the second the classical primal–dual affine–scaling method of Monteiro* et al. *[74]. The extended algorithms have the same worst-case complexity bounds as their LP counterparts. The failure of other related affine–scaling methods for SDP is discussed at the end of the chapter.*

3.1 Introduction

The introduction of Karmarkar's polynomial–time projective method for LP in 1984 [58] was accompanied by claims of some superior computational results. Later it seemed likely that the computation was done with a variant of the *affine–scaling* method, proposed by Dikin nearly two decades earlier in 1967 [29]. The two algorithms are closely related, and modifications of Karmarkar's algorithm by Vanderbei *et al.* [113] and Barnes [11] proved to be a rediscovery of the affine–scaling method. Dikin's affine–scaling method is a purely primal method, and the underlying idea is to minimize the objective function over an ellipsoid which is inscribed in the primal feasible region. Interestingly enough, polynomial com-

plexity of Dikin's affine–scaling method in its original form has still not been proved. Even more interesting is that the extension of this method to SDP may fail to converge to an optimal solution.[1]

In the primal–dual setting, the natural extension of the notion of affine–scaling is to minimize the duality gap over some inscribed ellipsoid in the primal–dual space. A primal–dual affine–scaling method is studied by Monteiro *et al.* in [74] where the primal–dual search direction minimizes the duality gap over a sphere in the primal–dual space. This algorithm may be viewed as a 'greedy' primal–dual algorithm, which aims to reach optimality in a single iteration, without attempting to stay centered. The worst–case iteration complexity for this method is $O(nL^2)$, where[2]

$$L := \log\left(\frac{\text{initial duality gap}}{\epsilon}\right).$$

As such, it is an algorithm of great theoretical interest. An understanding of the behavior of the method is essential in understanding the computationally efficient predictor–corrector methods, where the classical primal–dual affine–scaling direction is used as a predictor direction (see Chapters 4 and 6).

Jansen *et al.* proposed a primal–dual Dikin-type affine–scaling variant in [49] with improved $O(nL)$ polynomial complexity. This search direction minimizes the duality gap over the so-called Dikin ellipsoid in the primal–dual space. The interesting feature of this method is that each step involves both centering and reduction of the duality gap.

It was shown in [50] that the Dikin-type affine–scaling method [49] and the original primal-dual affine–scaling method [74] both belong to a generalized family of affine–scaling directions.

In this chapter *both* the primal–dual affine–scaling method of Monteiro *et al.* [74] and the method of Jansen *et al.* [49] are generalised to SDP. The former will be referred to as the *classical primal–dual affine–scaling method,* and the latter as the *primal–dual Dikin affine–scaling method* (or Dikin-type method). The Nesterov-Todd scaling (see Table 1.1) is used in defining the search directions; this is crucial, since the classical affine-scaling method fails for the scalings $S^{\frac{1}{2}}$ and I in

[1] See [78] and the bibliographical note (Section 3.6) at the end of this chapter.

[2] In LP, the value L equals the bitlength L_{bit}, say, of the input: if $\epsilon \leq 2^{-L_{bit}}$ then it is possible to round an ϵ-optimal solution pair (X^*, S^*) with $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$ to exact primal and dual solutions (see *e.g.* [98, 114]). In SDP, this association between L and L_{bit} is meaningless — there is no choice for ϵ which would yield exact optimal solutions. In fact, it is easy to construct instances of SDP problems with integer data but unique irrational solutions.

Table 1.1. These interesting negative results were recently proved in [79] (see also the bibliographical notes in Section 3.6 at the end of this chapter).

As discussed in the Chapter 1, the extension of interior point methods from LP to SDP is currently an active research area. Recently, the main focus has been done on central path following algorithms, see *e.g.* [31, 43] (primal methods), and [61, 73, 90, 106] (primal–dual methods). The methods presented in this chapter do not belong to any of these two classes, and as such constitute a different approach. In particular, a nearly centered starting solution is not required, although the worst case complexity bounds depend on the degree of centrality of the starting solution.

The importance of algorithms which can start from arbitrary feasible points is discussed by Goldfarb and Scheinberg in [37], where they study trajectories leading to the optimal set from arbitrary feasible starting points.

Outline of this chapter

The rest of this chapter is structured as follows. The Dikin-type method will be presented first, and its simple analysis will then be extended to the classical primal–dual affine–scaling method. Some preliminaries are discussed in the remainder of this section. In, particular, symmetric primal–dual scaling is discussed, and the algorithms are introduced. In Section 3.2 is shown how the two different primal–dual affine–scaling directions are derived by working in a scaled primal–dual space. It is shown how the two directions correspond to the minimization of the duality gap over two different ellipsoids in the scaled space. In Section 3.3 conditions to ensure a feasible steplength are derived, and the polynomial complexity result for the Dikin-type affine–scaling method are proven in Section 3.4. In Section 3.5 the analysis is extended to include the classical primal-dual affine–scaling method. Finally, the failure of some related affine–scaling methods for SDP are reviewed in Section 3.6.

3.1.1 Symmetric primal–dual scaling

For strictly feasible solutions $X \succ 0$ and $S \succ 0$ to (P) and (D) respectively, the scaling–matrix

$$D := S^{-\frac{1}{2}} \left(S^{\frac{1}{2}} X S^{\frac{1}{2}} \right)^{\frac{1}{2}} S^{-\frac{1}{2}}, \tag{3.1}$$

introduced in Section 1.6.1, satisfies $D^{-1}X = SD$, or

$$D^{-\frac{1}{2}}XD^{-\frac{1}{2}} = D^{\frac{1}{2}}SD^{\frac{1}{2}} := V.$$

In other words, the matrix D may be used to scale the variables X and S to the same symmetric positive definite matrix V. As mentioned in the Chapter 1, the matrix D was introduced by Nesterov and Todd in [82] and later by Sturm and Zhang [106] from a different perspective. We will sometimes refer to it as NT scaling, for short.[3]

Note that

$$V^2 = D^{-\frac{1}{2}}XSD^{\frac{1}{2}} \sim XS,$$

i.e. V^2 has the same eigenvalues as XS and is symmetric positive definite. As a consequence the duality gap is given by

$$\mathbf{Tr}\,(XS) = \mathbf{Tr}\,(V^2) = \|V\|^2 = \sum_i \lambda_i^2(V).$$

We can similarly scale any pair primal–dual search directions; feasible search directions $(\Delta X, \Delta y, \Delta S)$ must satisfy

$$\left.\begin{array}{rcl} \mathbf{Tr}\,(A_i\Delta X) & = & 0, \quad i = 1, \ldots, m \\ \sum_{i=1}^m \Delta y_i A_i + \Delta S & = & 0. \end{array}\right\} \tag{3.2}$$

Note that ΔX and ΔS are orthogonal, *i.e.* $\mathbf{Tr}\,(\Delta X \Delta S) = 0$.

The *scaled search directions* are defined by

$$D_X := D^{-\frac{1}{2}}\Delta X D^{-\frac{1}{2}}$$

and

$$D_S := D^{\frac{1}{2}}\Delta S D^{\frac{1}{2}}.$$

The scaled directions D_X and D_S are orthogonal by the orthogonality of ΔX and ΔS, *i.e.* $\mathbf{Tr}\,(D_X D_S) = 0$. Using (3.2), we obtain

$$D_S := -\sum_{i=1}^m \Delta y_i D^{\frac{1}{2}} A_i D^{\frac{1}{2}},$$

[3] In practice the scaling matrix D may be computed from the Choleski factorizations of X and S, and one additional singular value decomposition (SVD) (see Todd *et al.* [108]).

i.e. D_S must be in the span of matrices $D^{\frac{1}{2}}A_iD^{\frac{1}{2}}$ and D_X in its orthogonal complement, *i.e.*

$$\mathbf{Tr}\,\left(D^{\frac{1}{2}}A_iD^{\frac{1}{2}}D_X\right) = 0, \quad i = 1, \ldots, m.$$

The *scaled Newton step* is defined by

$$D_V := D_X + D_S.$$

After a feasible primal–dual step $(\Delta X, \Delta S)$ the duality gap becomes

$$\mathbf{Tr}\,((X + \Delta X)(S + \Delta S)) = \mathbf{Tr}\,((V + D_X)(V + D_S)) = \mathbf{Tr}\,(V^2 + VD_V), \tag{3.3}$$

where we have used the linearity of the trace as well as the property $\mathbf{Tr}\,(AB) = \mathbf{Tr}\,(BA)$.

3.1.2 Primal–dual affine–scaling search directions

The search direction of the Dikin–type affine–scaling algorithm minimizes the duality gap (3.3) over the so-called Dikin ellipsoid in the scaled primal–dual space, which will be defined in Section 3.2. We will see in Section 3.2 that the computation of this search direction amounts to the solution of

$$\Delta X + D\Delta SD = -\frac{XSX}{(\mathbf{Tr}\,(XS)^2)^{\frac{1}{2}}}, \tag{3.4}$$

subject to the conditions (3.2).

The classical primal–dual affine–scaling search direction minimizes the duality gap over a sphere; the computation of this direction involves the solution of the system

$$\Delta X + D\Delta SD = -X, \tag{3.5}$$

subject to (3.2), as will be shown in in Section 3.2. This direction is also well-known as the predictor direction in predictor–corrector methods using the NT scaling [106].

It is easily shown that (3.4) and (3.2) imply

$$\sum_{j=1}^{n} \Delta y_j \mathbf{Tr}\,(A_iDA_jD) = \frac{-\mathbf{Tr}\,(A_iXSX)}{(\mathbf{Tr}\,(XS)^2)^{\frac{1}{2}}}, \quad i = 1, \ldots, m, \tag{3.6}$$

for the Dikin-type step direction and that (3.5) and (3.2) imply

$$\sum_{j=1}^{n} \Delta y_j \mathbf{Tr}\ (A_i D A_j D) = -\mathbf{Tr}\ (A_i X)\,, \quad i = 1, \ldots, m, \tag{3.7}$$

for the classical primal-dual affine–scaling direction. The solution of these $m \times m$ linear systems yield Δy for the respective search directions. The coefficient matrices of the systems (3.6) and (3.7) are positive definite; a simple proof of this is given in Appendix A, Lemma A.2.2, based on a proof given by Faybusovich [30].

Once Δy is known, ΔS follows from $\sum_{i=1}^{m} \Delta y_i A_i = -\Delta S$, and ΔX is subsequently obtained from (3.4) (Dikin-type steps) or (3.5) (classical primal-dual affine–scaling steps).

Search directions which are determined by equations (3.2) and (3.4) (or an equation with a different right hand side like eq. (3.5)) can be computed in $mn^3 + \frac{1}{2}m^2n^2 + O(\max\{m,n\}^3)$ flops. The reader is referred to Monteiro and Zanjácomo [77] for details.

3.1.3 Measure of centrality

The Dikin-type steps have the feature that the proximity to the central path is maintained, where this proximity is quantified by

$$\kappa(XS) := \frac{\lambda_{\max}(XS)}{\lambda_{\min}(XS)} \tag{3.8}$$

with $\lambda_{\max}(XS)$ the largest eigenvalue of XS and $\lambda_{\min}(XS)$ the smallest. The classical affine–scaling steps may become increasingly less centered with respect to this measure, which complicates the analysis somewhat.

Note that $\kappa(XS) \geq 1$ and $\kappa(XS) = 1$ if and only if $XS = \mu I$ for some $\mu > 0$, *i.e.* if the pair (X, S) is centered with parameter μ.

Another frequently encountered centrality measure (see *e.g.* Sturm and Zhang [106]) is:

$$\bar{\delta}(XS) := \left\| I - \frac{1}{\mu} V^2 \right\|,$$

where $\mu = \mathbf{Tr}\,(XS)/n = \mathbf{Tr}\,(V^2)/n$. Note that one may have $\bar{\delta}(XS) = O(\sqrt{n})$ if $\kappa(XS) = O(1)$. This shows that $\kappa(XS) \leq \tau$ defines a larger neighbourhood

of the central path than $\delta(XS) \leq \tau$. The following relation between the two measures is readily proved:

$$\begin{aligned}
\bar{\delta}(XS) &= \left\| I - \frac{n}{\mathbf{Tr}\,(V^2)} V^2 \right\| \\
&\leq \sqrt{n} \left\| I - \frac{n}{\mathbf{Tr}\,(V^2)} V^2 \right\|_2 \\
&= \sqrt{n} \left(1 - \frac{n\lambda_{\min}(V^2)}{\sum_i \lambda_i(V^2)} \right) \\
&\leq \sqrt{n} \left(1 - \frac{\lambda_{\min}(V^2)}{\lambda_{\max}(V^2)} \right) \\
&= \sqrt{n} \left(1 - \frac{1}{\kappa(XS)} \right).
\end{aligned}$$

The relationship between κ and other centrality measures will be further explored in Chapter 4.

3.1.4 The Algorithms

The two primal–dual affine–scaling algorithms can both be described in the following framework:

Short step primal-dual affine-scaling algorithms

Input
A strictly feasible pair (X^0, S^0);

Parameters
$\tau_0 > 1$ such that $\kappa\,(X^0 S^0) \leq \tau_0$;
accuracy $\epsilon > 0$;
$L := \log \frac{\mathbf{Tr}\,(X^0 S^0)}{\epsilon}$;
$\alpha := \frac{1}{\sqrt{n}\tau_0}$ (Dikin-type steps), or
$\alpha := \frac{1}{nL\tau_0}$ (Classical affine–scaling steps);

begin
$X := X^0$; $S := S^0$;
while $\mathbf{Tr}\,(XS) > \epsilon$ **do**
Compute ΔX, ΔS from (3.4) and (3.2) (Dikin-type steps)
or from (3.5) and (3.2) (Classical affine–scaling steps);
$X := X + \alpha \Delta X$;
$S := S + \alpha \Delta S$;

end
end

We prove that the Dikin-type algorithm computes a strictly feasible ϵ-optimal solution (X^*, S^*) in $O(\tau_0 nL)$ steps, and this solution satisfies $\kappa(X^* S^*) \leq \tau_0$. The classical primal–dual affine–scaling algorithm converges in $O(\tau_0 nL^2)$ steps, and the solution satisfies $\kappa(X^* S^*) \leq 3\tau_0$.

3.2 Minimizing the duality gap over ellipsoids

The two primal–dual affine–scaling directions are defined as minimizing the duality gap over two different ellipsoids.

3.2.1 The Dikin-type step direction

The search direction of the primal–dual Dikin-type affine–scaling algorithm is derived by minimizing the duality gap over the so-called *Dikin ellipsoid*

$$D_V^* := \arg\min_{D_V} \left\{ \mathbf{Tr}\,(V^2 + VD_V) \;:\; \|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\| \leq 1 \right\}. \tag{3.9}$$

Note that $V + D_V \succeq 0$ if D_V is feasible in (3.9). It is easily verified that the optimal solution is given by

$$D_V^* = D_X^* + D_S^* = -\frac{V^3}{\|V^2\|}. \tag{3.10}$$

The transformation back to the unscaled space is done by premultiplying and post-multiplying (3.10) by $D^{\frac{1}{2}}$ to obtain

$$\Delta X + D\Delta S D = \frac{-XSX}{(\mathbf{Tr}\,(XS)^2)^{\frac{1}{2}}}. \tag{3.11}$$

The Dikin-type primal–dual affine–scaling direction is obtained by solving (3.11) subject to the conditions (3.2).

3.2.2 The classical primal-dual affine–scaling direction

The classical primal–dual affine–scaling direction is derived by minimizing the duality gap over a sphere:

$$D_V^* := \arg\min_{D_V} \left\{ \mathbf{Tr}\,(V^2 + VD_V) \;:\; \|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|_2 \leq 1 \right\}. \tag{3.12}$$

Note that $\|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|_2 \leq 1$ is equivalent to the condition

$$I \succeq V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}} \succeq -I.$$

This implies $V + D_V \succeq 0$ and $V - D_V \succeq 0$ which in turn implies that

$$\begin{aligned} 0 &\leq \mathbf{Tr}\,((V + D_V)(V - D_V)) \\ &= \|V\|^2 - \|D_V\|^2, \end{aligned}$$

or, $\|D_V\|^2 \leq \|V\|^2$ (spherical constraint). Using this reformulation it is easy to show that $D_V^* = -V$. Premultiplying and postmultiplying $D_V^* = -V$ by $D^{\frac{1}{2}}$ as before, one obtains

$$\Delta X + D\Delta S D = -X. \tag{3.13}$$

The solution of this equation subject to conditions (3.2) yields the classical primal-dual affine–scaling direction.

3.2.3 A note on the Dikin ellipsoid

There is some inconsistency in the literature concerning the definition of the primal–dual Dikin ellipsoid. In the paper by Nemirovskii and Gahinet [81] it is defined as

$$\|X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}\|^2 + \|S^{-\frac{1}{2}}\Delta S S^{-\frac{1}{2}}\|^2 \leq 1, \tag{3.14}$$

which is the same as

$$\|V^{-\frac{1}{2}}D_X V^{-\frac{1}{2}}\|^2 + \|V^{-\frac{1}{2}}D_S V^{-\frac{1}{2}}\|^2 \leq 1.$$

A primal–dual step $(X + \Delta X, S + \Delta S)$ which satisfies (3.14) is always feasible: From (3.14) one has $-1 \leq \lambda_i \left(X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}\right) \leq 1$ $(i = 1, \ldots, n)$, or

$$X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}} \succeq -I,$$

which in turn implies $X + \Delta X \succeq 0$. Similarly one has $S + \Delta S \succeq 0$.

The relation between the ellipsoid defined by (3.14) and the primal–dual Dikin ellipsoid in (3.12) is shown in the following lemma.

Lemma 3.2.1 *Let* $\varepsilon_D := \|V^{-\frac{1}{2}}D_V V^{-\frac{1}{2}}\|^2$ *and* $\varepsilon_{NG} := \|V^{-\frac{1}{2}}D_X V^{-\frac{1}{2}}\|^2 + \|V^{-\frac{1}{2}}D_S V^{-\frac{1}{2}}\|^2$. *One has*

$$\frac{1}{\kappa(V^2)}\varepsilon_D \leq \varepsilon_{NG} \leq \kappa(V^2)\varepsilon_D.$$

Proof:

In the proof we will repeatedly use the easily proven inequality

$$\mathbf{Tr}\,(AB) \leq \lambda_{\max}(A)\mathbf{Tr}\,(B), \quad \text{for } A, B \succeq 0, \tag{3.15}$$

which is equivalent to[4] the well-known inequality

$$\|AB\| \leq \|A\|_2 \|B\|, \quad \text{for } A, B \in \mathbb{R}^{n\times n}. \tag{3.16}$$

By definition of the ellipsoid in (3.14):

$$\begin{aligned}
\varepsilon_{NG} &= \|V^{-\frac{1}{2}} D_X V^{-\frac{1}{2}}\|^2 + \|V^{-\frac{1}{2}} D_S V^{-\frac{1}{2}}\|^2 \\
&= \mathbf{Tr}\left(V^{-1} D_X V^{-1} D_X + V^{-1} D_S V^{-1} D_S\right) \\
&\leq \lambda_{\max}(V^{-2}) \mathbf{Tr}\left(D_X^2 + D_S^2\right) \\
&= \frac{1}{\lambda_{\min}(V^2)} \|D_V\|^2 \\
&\leq \frac{\lambda_{\max}(V^2)}{\lambda_{\min}(V^2)} \|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|^2 = \kappa(V^2)\varepsilon_D,
\end{aligned}$$

where we have used (3.15) and (3.16) successively, as well as the orthogonality of D_X and D_S. Conversely,

$$\begin{aligned}
\varepsilon_D &= \|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|^2 \\
&= \mathbf{Tr}\left(V^{-1} D_V V^{-1} D_V\right) \\
&\leq \lambda_{\max}\left(V^{-2}\right) \mathbf{Tr}\left(D_V^2\right) \\
&= \frac{1}{\lambda_{\min}(V^2)} \mathbf{Tr}\left(D_X^2 + D_S^2\right) \\
&\leq \frac{\lambda_{\max}(V^2)}{\lambda_{\min}(V^2)} \mathbf{Tr}\left(V^{-1} D_X V^{-1} D_X + V^{-1} D_S V^{-1} D_S\right) \\
&= \kappa(V^2)\left(\left\|V^{-\frac{1}{2}} D_X V^{-\frac{1}{2}}\right\|^2 + \left\|V^{-\frac{1}{2}} D_S V^{-\frac{1}{2}}\right\|^2\right) = \kappa(V^2)\varepsilon_{NG}.
\end{aligned}$$

The result follows. □

A corollary to this lemma is that the two ellipsoids $\varepsilon_D \leq 1$ and $\varepsilon_{NG} \leq 1$ coincide on the central path (where $\kappa(V^2) = 1$).

[4]The equivalence follows by replacing A by AA^T and B by BB^T in (3.15).

3.3 Feasibility of the Dikin-type step

We proceed with the analysis of the Dikin-type affine–scaling method, after which we will extend the analysis to cover the classical primal-dual affine–scaling method.

Having computed the Dikin-type step direction $(\Delta X, \Delta S)$ from (3.11) and (3.2), a feasible steplength must be established. Denoting

$$X(\alpha) := X + \alpha \Delta X, \quad S(\alpha) := S + \alpha \Delta S,$$

we establish a value $\bar{\alpha} > 0$ such that $X(\bar{\alpha}) \succ 0$ and $S(\bar{\alpha}) \succ 0$. The following lemma gives a sufficient condition for a feasible steplength $\bar{\alpha}$.

Lemma 3.3.1 *Let $X \succ 0$ and $S \succ 0$. If one has*

$$\det\left(X(\alpha)S(\alpha)\right) > 0 \quad \forall\, 0 \leq \alpha \leq \bar{\alpha},$$

then $X(\bar{\alpha}) \succ 0$ and $S(\bar{\alpha}) \succ 0$.

Proof:

The function

$$f(\alpha, \lambda) := \det\left[(X + \alpha \Delta X) - \lambda I\right]$$

is continuously differentiable if $\alpha \in (0, \bar{\alpha})$ and $\lambda \in \mathbb{R}$, and is zero if λ is an eigenvalue of $X(\alpha)$. The implicit function theorem therefore implies that the eigenvalues of $X(\alpha)$ (and similarly of $S(\alpha)$) are continuous functions of α.

Since $\det(X(\alpha)S(\alpha)) = \det(X(\alpha))\det(S(\alpha))$, one has

$$\det(X(\alpha)S(\alpha)) = \prod_i \lambda_i(X(\alpha)) \prod_i \lambda_i(S(\alpha)). \tag{3.17}$$

The left hand side of eq. (3.17) is strictly positive on $[0, \bar{\alpha}]$. This shows that the eigenvalues of $X(\alpha)$ and $S(\alpha)$ remain positive on $[0, \bar{\alpha}]$. □

In order to derive bounds on α which are sufficient to guarantee a feasible steplength, we need the following three technical results.

Lemma 3.3.2 *The spectral radius of $D_X D_S + D_S D_X$ is bounded by*

$$\rho(D_X D_S + D_S D_X) \leq \frac{1}{2}\|D_X + D_S\|^2.$$

Proof:

It is trivial to verify that

$$D_X D_S + D_S D_X = \frac{1}{2}\left[(D_X + D_S)^2 - (D_X - D_S)^2\right]$$

which implies

$$-\frac{1}{2}(D_X - D_S)^2 \preceq D_X D_S + D_S D_X \preceq \frac{1}{2}(D_X + D_S)^2.$$

It follows that

$$-\frac{1}{2}\|D_X - D_S\|^2 I \preceq D_X D_S + D_S D_X \preceq \frac{1}{2}\|D_X + D_S\|^2 I.$$

Since D_X and D_S are orthogonal the matrices $(D_X + D_S)$ and $(D_X - D_S)$ have the same norm. Consequently

$$-\frac{1}{2}\|D_X + D_S\|^2 I \preceq D_X D_S + D_S D_X \preceq \frac{1}{2}\|D_X + D_S\|^2 I$$

from which the required result follows. □

Corollary 3.3.1 *For the Dikin-type step $D_X + D_S = -V^3/\|V^2\|$, one has*

$$\rho(D_X D_S + D_S D_X) \leq \frac{1}{2}\rho(V^2).$$

Proof:

By Lemma 3.3.2 one has

$$\begin{aligned} 2\rho\left(D_X D_S + D_S D_X\right) &\leq \|D_X + D_S\|^2 \\ &= \left(\frac{\|V^3\|}{\|V^2\|}\right)^2 = \frac{\mathbf{Tr}\left(V^6\right)}{(\|V^2\|)^2} \\ &\leq \rho(V^2)\frac{\mathbf{Tr}\left(V^4\right)}{(\|V^2\|)^2} = \rho(V^2), \end{aligned}$$

which is the required result. □

The following lemma contains two useful results from linear algebra concerning semidefinite matrices. It is proved in Appendix A.

Lemma 3.3.3 *Let $Q \succ 0$, and let $M \in \mathbb{R}^{n\times n}$ be skew–symmetric. One has $\det(Q+M) > 0$. Moreover, if it is known that $\lambda_i(Q+M) \in \mathbb{R}$ $(i = 1,\ldots,n)$, then*

$$0 < \lambda_{\min}(Q) \leq \lambda_{\min}(Q+M) \leq \lambda_{\max}(Q+M) \leq \lambda_{\max}(Q),$$

which implies $\kappa(Q+M) \leq \kappa(Q)$.

We are now in a position to find a step size α which guarantees that the Dikin-type step will be feasible. To simplify the analysis we introduce a parameter $\tau > 1$ such that $\kappa(XS) = \kappa(V^2) \leq \tau$. This implies the existence of numbers τ_1 and τ_2 such that

$$\tau_1 I \preceq V^2 \preceq \tau_2 I, \quad \tau_2 = \tau_1\tau. \tag{3.18}$$

Lemma 3.3.4 *The steps $X(\alpha) = X + \alpha\Delta X$ and $S(\alpha) = S + \alpha\Delta S$ are feasible if the step size α satisfies $\alpha \leq \bar{\alpha}$ where*

$$\bar{\alpha} = \min\left\{\frac{\|V^2\|}{2\tau_2}, \frac{4\tau_1}{\|V^2\|}\right\}.$$

Furthermore

$$\kappa\left(X(\bar{\alpha})S(\bar{\alpha})\right) \leq \tau.$$

Proof:

We show that the determinant of $X(\alpha)S(\alpha)$ remains positive for all $\alpha \leq \bar{\alpha}$. One then has $X(\bar{\alpha}), S(\bar{\alpha}) \succ 0$ by Lemma 3.3.1.

To this end note that

$$\begin{aligned}
X(\alpha)S(\alpha) &\sim (V+\alpha D_X)(V+\alpha D_S) \\
&= V^2 + \alpha D_X V + \alpha V D_S + \alpha^2 D_X D_S \\
&= V^2 - \frac{\alpha V^4}{\|V^2\|} + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X) \\
&\quad + \left[\frac{1}{2}\alpha^2(D_X D_S - D_S D_X) + \frac{1}{2}\alpha(D_X V + V D_S - V D_X - D_S V)\right],
\end{aligned}$$

since $D_X + D_S = -V^3/\|V^2\|$. The matrix in square brackets is skew–symmetric. Lemma 3.3.3 therefore implies that the determinant of $[X(\alpha)S(\alpha)]$ will be positive if the matrix

$$Q(\alpha) := V^2 - \frac{\alpha V^4}{\|V^2\|} + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X)$$

is positive definite. Note that $Q(0) = V^2 \succ 0$ and $\kappa\left(Q(0)\right) \leq \tau$. We proceed to prove that $\kappa\left(Q(\alpha)\right)$ remains bounded by $\kappa(Q(\alpha)) \leq \tau$ for $0 \leq \alpha \leq \bar{\alpha}$. This is sufficient to prove that $Q(\alpha) \succ 0,\ 0 \leq \alpha \leq \bar{\alpha}$, and therefore that a step of length $\bar{\alpha}$ is feasible.

Moreover, after such a feasible step we will have $X(\bar{\alpha}) \succ 0$, $S(\bar{\alpha}) \succ 0$. The matrix $X(\bar{\alpha})S(\bar{\alpha})$ therefore has positive eigenvalues and we can apply the second part of Lemma 3.3.3 to obtain

$$\kappa\left(X(\bar{\alpha})S(\bar{\alpha})\right) \leq \kappa\left(Q(\bar{\alpha})\right) \leq \tau.$$

We start the proof by noting that if λ is an eigenvalue of V^2 then $(\lambda - \alpha\lambda^2/\|V^2\|)$ is an eigenvalue of $[V^2 - \alpha V^4/\|V^2\|]$. The function

$$\phi(t) := t - \alpha\frac{t^2}{\|V^2\|}$$

is monotonically increasing on $t \in [0, \tau_2]$ if $\alpha \leq \bar{\alpha}$, since $\bar{\alpha} \leq \|V^2\|/(2\tau_2)$. Thus

$$\phi(\tau_1)I \preceq V^2 - \frac{\alpha V^4}{\|V^2\|} \preceq \phi(\tau_2)I \quad \forall\ 0 \leq \alpha \leq \bar{\alpha}$$

or, equivalently,

$$\phi(\tau_1)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X) \preceq Q(\alpha) \preceq \phi(\tau_2)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X) \quad \forall\ 0 \leq \alpha \leq \bar{\alpha}.$$

We will therefore certainly have $\kappa(Q(\alpha)) \leq \tau$ if

$$\tau[\phi(\tau_1)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X)] \succeq \phi(\tau_2)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X).$$

This matrix inequality can be simplified using $\tau_2 = \tau\tau_1$ and subsequently dividing by α. This yields

$$\left(\frac{\tau_2^2 - \tau\tau_1^2}{\|V^2\|}\right) I + \alpha(\tau - 1)\left(\frac{1}{2}(D_X D_S + D_S D_X)\right) \succeq 0. \tag{3.19}$$

Expression (3.19) may be further simplified using

$$\tau_2^2 - \tau\tau_1^2 = (\tau - 1)\tau_1\tau_2$$

to obtain

$$\left(\frac{\tau_1\tau_2}{\|V^2\|}\right) I + \frac{1}{2}\alpha(D_X D_S + D_S D_X) \succeq 0$$

which will surely hold if

$$\left(\frac{\tau_1\tau_2}{\|V^2\|}\right) I - \alpha\rho\left(\frac{1}{2}(D_X D_S + D_S D_X)\right) I \succeq 0.$$

Substituting the bound

$$\rho\left(\frac{1}{2}(D_X D_S + D_S D_X)\right) \leq \frac{1}{4}\rho(V^2) \leq \frac{1}{4}\tau_2$$

from Corollary 3.3.1 yields

$$\frac{\tau_1\tau_2}{\|V^2\|} - \frac{1}{4}\alpha\tau_2 \geq 0,$$

or

$$\alpha \leq \frac{4\tau_1}{\|V^2\|},$$

which is the second bound in the lemma. □

3.4 Convergence and complexity analysis

A feasible Dikin-type step of length α reduces the duality gap by at least a factor $(1 - \frac{\alpha}{\sqrt{n}})$. Formally we have the following result.

Lemma 3.4.1 *Given a feasible primal–dual pair (X, S) and a steplength α such that the Dikin-type step is feasible,* i.e. $X(\alpha) := X + \alpha\Delta X \succ 0$*, and $S(\alpha) := S + \alpha\Delta S \succ 0$, it holds that*

$$\mathbf{Tr}\ (X(\alpha)S(\alpha)) \leq \left(1 - \frac{\alpha}{\sqrt{n}}\right) \mathbf{Tr}\,(XS).$$

Proof:

The duality gap after the Dikin-type step is given by

$$\begin{aligned}
\mathbf{Tr}\ (X(\alpha)S(\alpha)) &= \mathbf{Tr}\ ((V+\alpha D_X)(V+\alpha D_S)) \\
&= \mathbf{Tr}\ \left(V^2+\alpha V(D_X+D_S)\right) \\
&= \mathbf{Tr}\ \left(V^2-\alpha\frac{V^4}{\|V^2\|}\right) \\
&= \|V\|^2-\alpha\|V^2\| \\
&= \left(1-\alpha\frac{\|V^2\|}{\|V\|^2}\right)\mathbf{Tr}\,(XS).
\end{aligned}$$

By the Cauchy-Schwartz inequality one has

$$\|V\|^2=\mathbf{Tr}\ \left(IV^2\right)\leq\|I\|\left\|V^2\right\|=\sqrt{n}\left\|V^2\right\|,$$

which gives the required result. □

We are now ready to prove a worst-case iteration complexity bound.

Theorem 3.4.1 *Let $\epsilon>0$ be an accuracy parameter, and let $\tau_0>1$ be such that $\kappa(X^0S^0)\leq\tau_0$. Further let $L=\log\left(\mathbf{Tr}\,(X^0S^0)/\epsilon\right)$, and $\alpha=\frac{1}{\tau_0\sqrt{n}}$. The Dikin-type step algorithm requires at most $\lceil\tau_0 nL\rceil$ iterations to compute a feasible primal–dual pair (X^*,S^*) satisfying $\kappa(X^*S^*)\leq\tau_0$ and $\mathbf{Tr}\,(X^*S^*)\leq\epsilon$.*

Proof:

We first prove that the default choice of α always allows a feasible step. To this end, note that

$$\alpha=\frac{1}{\tau_0\sqrt{n}}=\frac{\tau_1}{\tau_2\sqrt{n}}\leq\frac{\tau_1\sqrt{n}}{2\tau_2}=\frac{\|\tau_1 I\|}{2\tau_2}\leq\frac{\|V^2\|}{2\tau_2},$$

since $0\preceq\tau_1 I\preceq V^2$. This shows that α meets the first condition of Lemma 3.3.4. Moreover, it holds that $\|V^2\|\leq\tau_2\sqrt{n}$, which implies

$$\frac{4\tau_1}{\|V^2\|}\geq\frac{4\tau_1}{\tau_2\sqrt{n}}=\frac{4}{\tau_0\sqrt{n}}>\alpha.$$

The default choice of α therefore meets the conditions of Lemma 3.3.4 and ensures a feasible Dikin-type step.

We know by Lemma 3.4.1 that the duality gap is reduced at each iteration by at least a factor $(1 - \frac{1}{n\tau_0})$. As the initial duality gap equals $\mathbf{Tr}\,(X^0S^0)$, the duality gap will be smaller than ϵ after k iterations if

$$\left(1 - \frac{1}{n\tau_0}\right)^k \mathbf{Tr}\,(X^0S^0) \leq \epsilon.$$

Taking logarithms yields

$$k \log\left(1 - \frac{1}{n\tau_0}\right) + \log\left(\mathbf{Tr}\,(X^0S^0)\right) \leq \log(\epsilon). \tag{3.20}$$

Since

$$-\log\left(1 - \frac{1}{n\tau_0}\right) \geq \left(\frac{1}{n\tau_0}\right),$$

condition (3.20) will certainly be satisfied if

$$\frac{k}{n\tau_0} \geq \log\left(\mathbf{Tr}\,(X^0S^0)\right) - \log\epsilon = L,$$

which implies the required result. □

The $O(\tau_0 n)$ complexity bound is a factor $\sqrt{n}$ worse than the best known bound for primal–dual algorithms, but this is due to the use of large neighbourhoods of the central path. It will be shown in the next chapter that the complexity can be improved to $O(\sqrt{\tau_0 n})$ if longer steps are used in conjunction with 'centering steps', *i.e.* steps to improve centrality. This latter bound is the best complexity bound known for primal–dual methods.

3.5 The classical primal–dual affine–scaling method

We return to the analysis of the classical primal–dual affine–scaling algorithm. This analysis is analogous to that of the Dikin-type step method, but there is one significant difference: whereas the Dikin-type steps stay in the same neighbourhood of the central path, the same is not true of the classical affine–scaling steps.

The deviation from centrality increases at each step, but this can be bounded, and polynomial complexity can be retained at a price: The step length has to be shortened to

$$\alpha = \frac{1}{nL\tau_0}, \tag{3.21}$$

and the worst case iteration complexity bound becomes $O(\tau_0 nL^2)$.

We need to modify the analysis of the Dikin-type step algorithm with regard to the following:

- We allow for an increase in the distance $\kappa(XS)$ from the central path by a constant factor $t > 1$ at each step;
- The steplength α in (3.21) is shown to be feasible for $\tau_0 nL^2$ iterations, provided that we choose the factor t in such a way that the distance from the central path stays within the bound $\kappa(XS) < 3\tau_0$ for $O(\tau nL^2)$ iterations – the convergence criterion is met before the deviation from centrality becomes worse than $3\tau_0$.

Recall that the classical primal-dual affine–scaling direction is obtained by solving

$$\Delta X + D\Delta S D = -X,$$

subject to (3.2). A feasible step in this direction gives the following reduction in the duality gap:

Lemma 3.5.1 *Given a feasible primal–dual pair (X, S) and assume that the affine–scaling step with steplength α is feasible,* i.e. $X(\alpha) := X + \alpha\Delta X \succ 0$, *and $S(\alpha) := S + \alpha\Delta S \succ 0$. It holds that*

$$\mathbf{Tr}\ (X(\alpha)S(\alpha)) \leq (1-\alpha)\mathbf{Tr}\,(XS).$$

Proof:

Analogous to the proof of Lemma 3.4.1. □

As with the Dikin-type step analysis, we will also need the following bound:

Lemma 3.5.2 *For the primal–dual affine–scaling step $D_V = D_X + D_S = -V$, one has*

$$\rho(D_X D_S + D_S D_X) \leq \frac{1}{2}\|V\|^2.$$

Proof:

Follows from Lemma 3.3.2. □

Now let $\tau = \kappa(XS)$ and $\tau_0 = \kappa(X^0 S^0)$ for the current pair of iterates (X, S) and starting solution (X^0, S^0) respectively, and let τ_1, τ_2 satisfy (3.18).[5]

We also define the amplification factor

$$t := 1 + \frac{1}{nL^2\tau_0},$$

which is used to bound the deviation from centrality in a given iteration.

Lemma 3.5.3 *If $\tau \leq \frac{3\tau_0}{t}$, then the steps $X(\alpha) = X + \alpha\Delta X$ and $S(\alpha) = S + \alpha\Delta S$ are feasible for the step size*

$$\bar{\alpha} = \frac{1}{nL\tau_0},$$

and the deviation from centrality is bounded by

$$\kappa\left(X(\bar{\alpha})S(\bar{\alpha})\right) \leq t\tau.$$

Proof:

As in the proof of Lemma 3.3.4, we show that the determinant of $X(\alpha)S(\alpha)$ remains positive for all $\alpha \leq \bar{\alpha}$, which ensures $X(\bar{\alpha}), S(\bar{\alpha}) \succ 0$ by Lemma 3.3.1.

As before, note that

$$\begin{aligned} X(\alpha)S(\alpha) &\sim (V + \alpha D_X)(V + \alpha D_S) \\ &= V^2 + \alpha D_X V + \alpha V D_S + \alpha^2 D_X D_S \\ &= (1-\alpha)V^2 + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X) \\ &\quad + \left[\frac{1}{2}\alpha^2(D_X D_S - D_S D_X) + \frac{1}{2}\alpha(D_X V + V D_S - V D_X - D_S V)\right], \end{aligned}$$

[5] The value τ_0 had to be strictly greater than one, *i.e.* $\tau_0 > 1$, for the Dikin-type algorithm. Here it is sufficient to require $\tau_0 \geq 1$. The value $\tau_0 = \kappa(X^0 S^0)$ can therefore be used for the classical primal–dual affine–scaling method.

since $D_X + D_S = -V$. The matrix in square brackets is skew–symmetric. Lemma 3.3.3 therefore implies that the determinant of $[X(\alpha)S(\alpha)]$ will be positive if the matrix

$$Q(\alpha) := (1-\alpha)V^2 + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X)$$

is positive definite. Note that $Q(0) = V^2 \succ 0$ and $\kappa\left(Q(0)\right) = \tau$. We proceed to prove that $\kappa\left(Q(\alpha)\right)$ remains bounded by $\kappa(Q(\alpha)) \leq t\tau$ for $0 \leq \alpha \leq \bar{\alpha}$, for the fixed amplification factor t. This is sufficient to prove that $Q(\alpha) \succ 0,\ 0 \leq \alpha \leq \bar{\alpha}$, and therefore that a step of length $\bar{\alpha}$ is feasible.

Moreover, after such a feasible step we will have $X(\bar{\alpha}) \succ 0$, $S(\bar{\alpha}) \succ 0$. The matrix $X(\bar{\alpha})S(\bar{\alpha})$ therefore has positive eigenvalues and we can apply the second part of Lemma 3.3.3 to obtain

$$\kappa\left(X(\bar{\alpha})S(\bar{\alpha})\right) \leq \kappa\left(Q(\bar{\alpha})\right) \leq t\tau.$$

To start the proof, note that for all $0 \leq \alpha \leq \bar{\alpha}$ one has

$$\tau_1(1-\alpha)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X) \preceq Q(\alpha) \preceq \tau_2(1-\alpha)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X).$$

We will therefore certainly have $\kappa(Q(\alpha)) \leq t\tau$ if

$$t\tau[\tau_1(1-\alpha)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X)] \succeq \tau_2(1-\alpha)I + \frac{1}{2}\alpha^2(D_X D_S + D_S D_X).$$

Using $\tau_2 = \tau\tau_1$ the last relation becomes

$$\tau_2(1-\alpha)(t-1)I + \frac{1}{2}\alpha^2(t\tau - 1)(D_X D_S + D_S D_X) \succeq 0. \tag{3.22}$$

Since one has $\rho(D_X D_S + D_S D_X) \leq \frac{1}{2}\|V\|^2 \leq \frac{1}{2}\tau_2 n$ by Lemma 3.5.2, inequality (3.22) will hold if

$$(1-\alpha)(t-1) - \frac{1}{4}\alpha^2(t\tau - 1)n \geq 0. \tag{3.23}$$

Using the assumption $t\tau \leq 3\tau_0$, it follows that (3.23) will surely hold if

$$(1-\alpha)\left(\frac{1}{nL^2\tau_0}\right) - \frac{1}{4}\alpha^2(3\tau_0 - 1)n \geq 0,$$

which is satisfied by $\bar{\alpha} = \frac{1}{nL\tau_0}$. □

We now investigate how many iterations can be performed while still satisfying the assumption $\kappa(XS) \leq 3\tau_0/t$ of Lemma 3.5.3.

Lemma 3.5.4 *One has*

$$\kappa(XS) \leq 3\tau_0$$

for the first $\lceil nL^2\tau_0 \rceil$ *iterations of the classical primal–dual affine–scaling algorithm.*

Proof:

By Lemma 3.5.3 one has

$$\kappa(XS) \leq \tau_0 t^k \text{ after } k \text{ iterations,}$$

provided that k is sufficiently small to guarantee $\tau_0 t^k \leq 3\tau_0$. Using $t = 1 + \frac{1}{nL^2\tau_0}$, we obtain

$$t^k = \left(1 + \frac{1}{nL^2\tau_0}\right)^k < 3 \text{ if } k \leq \lceil nL^2\tau_0 \rceil,$$

which gives the required result. □

It only remains to prove that $\lceil nL^2\tau_0 \rceil$ iterations are sufficient to guarantee convergence. The proof is analogous to that of Theorem 3.4.1. Formally we have the following result:

Theorem 3.5.1 *Let* $\epsilon > 0$ *be an accuracy parameter, and let* τ_0 *be such that* $\kappa(X^0S^0) \leq \tau_0$. *Further let* $L = \log\left(\mathbf{Tr}\left(X^0S^0\right)/\epsilon\right)$ *and* $\alpha = \frac{1}{nL\tau_0}$. *The classical primal–dual affine–scaling algorithm requires at most* $\lceil nL^2\tau_0 \rceil$ *iterations to compute a feasible primal–dual pair* (X^*, S^*) *satisfying* $\kappa(X^*S^*) \leq 3\tau_0$ *and* $\mathbf{Tr}\left(X^*S^*\right) \leq \epsilon$.

Proof:

By Lemma 3.5.3 a step of size $\alpha = \frac{1}{nL\tau_0}$ is feasible as long as the iterates (X, S) satisfy $\kappa(XS) \leq 3\tau_0$. Such a step reduces the duality gap by a factor $(1 - \frac{1}{nL\tau_0})$ (Lemma 3.5.1).

By the proof of Theorem 3.4.1, this reduction of the duality gap ensures that the convergence criterium is met after k steps, if

$$k\alpha \geq \log \frac{\mathbf{Tr}\left(X^0S^0\right)}{\epsilon} = L,$$

i.e. if $k \geq nL^2\tau_0$. Lemma 3.5.4 guarantees that the first $nL^2\tau_0$ steps will be feasible, which completes the proof. □

3.6 Bibliographical notes: failure of related methods

The extensions of primal–dual affine–scaling methods from LP to SDP in this chapter are the only successful extensions of affine–scaling methods at the time of writing. Several negative results were obtained recently, *i.e.* other affine–scaling methods for SDP which are not globally convergent. These results are reviewed below.

3.6.1 Primal affine–scaling

The primal affine–scaling direction for semidefinite programming is derived by minimizing the primal objective over the *primal Dikin ellipsoid*:

$$\Delta X := \arg\min_{\Delta X} \left\{ \mathbf{Tr}\,(C\Delta X) \;:\; \left\| X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \right\|^2 \leq 1,\ \mathbf{Tr}\,(A_i \Delta X) = 0\ (i = 1, \ldots, m) \right\}. \tag{3.24}$$

Note that the step ΔX is always feasible.

Recently Muramatsu [78] proved that an algorithm using these steps can converge to a non-optimal point, regardless of which step length is used.

This surprising result gives added interest to the analysis of the primal-dual variants discussed in this chapter, and shows that the extension of algorithms from LP to SDP cannot always be taken for granted.

The primal affine–scaling search direction is in fact introduced in a different way in [78]; the so-called dual estimates $(y(X), S(X))$ are first computed via

$$(y(X), S(X)) := \arg\min_{y,S} \left\{ \left\| X^{\frac{1}{2}} S X^{\frac{1}{2}} \right\| \;:\; S + \sum_{i=1}^{m} y_i A_i = C \right\}, \tag{3.25}$$

after which the primal affine–scaling search direction is defined as

$$\Delta X := -XS(X)X. \tag{3.26}$$

The two definitions (3.24) and (3.26) yield the same direction (up to a constant scaling). This is easily proven by comparing the optimality conditions of the two minimization problems (3.24) and (3.25).

3.6.2 Primal–dual affine–scaling

The classical primal–dual affine–scaling direction analysed in this chapter is one of a family of such directions, which are determined by the choice of scaling matrix (see Section 1.6.2). The family of directions at a strictly feasible pair (X, S) is defined as the solutions of the system:

$$\begin{aligned} \mathbf{Tr}\,(A_i \Delta X) &= 0, \quad i = 1, \ldots, m \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S &= 0 \\ H_P\,(\Delta X S + X \Delta S) &= -H_P(XS), \end{aligned}$$

where H_P is the linear transformation given by

$$H_P(M) := \frac{1}{2}\left[PMP^{-1} + P^{-T}M^T P^T\right],$$

for any $M \in \mathcal{S}_n$, and P is one of the scaling matrices from Table 1.1.

The direction resulting from $P = D^{\frac{1}{2}}$ was analysed in this chapter (Section 3.5). The choices $P = S^{\frac{1}{2}}$ and $P = I$, however, do *not* allow convergent algorithms — Muramatsu and Vanderbei [79] recently provided the following example where the primal–dual affine–scaling algorithms using these directions fail to converge to the optimal solution:

$$\min\left\{\mathbf{Tr}\,(CX) \;:\; \mathbf{Tr}\,(AX) = 2,\; X \succeq 0\right\},$$

where

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The search direction corresponding to $P = S^{\frac{1}{2}}$ was independently proposed by Monteiro [73], Helmberg *et al.* [44], and Kojima *et al.* [65], and is sometimes referred to as the HRVW/KSH/M direction in the literature. It is interesting to note that the HRVW/KSH/M direction may also be derived by first solving

$$\begin{aligned} \mathbf{Tr}\left(A_i \widehat{\Delta X}\right) &= 0, \quad i = 1, \ldots, m \\ \textstyle\sum_{i=1}^{m} \Delta y_i A_i + \Delta S &= 0 \\ X\Delta S + \widehat{\Delta X} S &= -XS, \end{aligned} \tag{3.27}$$

and subsequently replacing $\widehat{\Delta X}$ (which is not necessarily symmetric) by its symmetric part $\Delta X = \frac{1}{2}\left(\widehat{\Delta X} + \widehat{\Delta X}^T\right)$. This makes the HRVW/KSH/M direction computationally attractive — no square root factorizations (or singular value decompositions) are needed.

The direction corresponding to $P = I$ is called the AHO direction, as it was introduced by Alizadeh, Haeberley and Overton [4].

Interestingly enough, the classical primal–dual affine scaling method using the NT direction converges for the above example, even if full Newton steps are used. This may even be true in general (see the discussion in [79]).

These results are important, since primal–dual affine–scaling directions are used as the predictor direction in predictor–corrector methods (see Section 1.6.1). The abovementioned failures are consistent with the finding that methods based on the NT direction are more robust than those using the HRVW/KSH/M or AHO directions, where jamming sometimes occurs [108].

Chapter 4

Primal–dual potential reduction methods

In Chapter 3, primal–dual affine–scaling methods were extended from linear programming to semidefinite programming. In this chapter it is shown how to analyse these methods in the framework of potential reduction algorithms. The analysis suggests implementable variants of the methods as 'predictor–corrector' type potential reduction algorithms. A numerical comparison with the potential reduction method of Nesterov and Todd is presented, where the new methods perform competitively.

4.1 Introduction

Potential reduction methods were the first methods to be extended from linear programming (LP) to the more general semidefinite programming (SDP) problem. In their seminal work [84] Nesterov and Nemirovskii present three methods, namely a generalization of Karmarkar's method [58], a projective method, and an extension of an LP algorithm of Ye [116]. Alizadeh [1] independently analysed several methods which have analogies in the LP literature.

A primal–dual potential reduction method suited for the structure of linear matrix inequalities arising in control theory applications was analysed by Vandenberghe and Boyd in [111]. A general potential reduction method for convex optimiza-

tion problems involving homogeneous self–dual cones (which includes SDP) is presented by Nesterov and Todd [85].

Most of these methods (and some variants thereof) are described in the review paper by Vandenberghe and Boyd [112]. A more recent survey of potential reduction methods is given by Anstreicher in [9]; this latter survey focuses on the LP case but includes extensions to linear complementarity (LCP) problems and conic convex programming.

In Chapter 3 the primal–dual affine–scaling method of Monteiro *et al.* [74] and the Dikin-type primal–dual affine–scaling method of Jansen *et al.* [49] were extended from LP to SDP. These methods are both short-step methods, and therefore not suitable for implementation.

The aim of this chapter is twofold. In the first instance it provides a new proof of the polynomial complexity of the short step methods in Chapter 3. In doing so, the analysis of the primal–dual affine–scaling methods is linked to that of primal–dual potential reduction methods. In the second instance, the analysis is used to formulate implementable versions of the primal–dual affine–scaling methods as predictor–corrector type potential reduction methods. It is shown that the new algorithms have $O(\sqrt{n}L)$ worst-case iteration complexity. A numerical comparison is made with the Nesterov–Todd potential reduction method, as implemented by Vandenberghe and Boyd [110]. The new algorithms perform competitively on a set of medium sized test problems.

Outline of this chapter

In order to keep the presentation simple, only the analysis for the primal–dual Dikin–type affine–scaling direction is presented. The analogous results for the classical primal–dual affine–scaling method can be derived in the same way, and will be listed in the last section of this chapter.

The centering effect of primal–dual Dikin steps is discussed in Section 4.2. The complexity analysis of the short step method is done in a potential reduction setting in Section 4.3. Longer steps via plane search of the potential function are reviewed in Section 4.4. Longer steps require some recentering afterwards, and the recentering process in described in Section 4.5 and Section 4.6. The complexity analysis of the resulting ‘predictor–corrector’ type method is done in Section 4.7. Numerical results are given in Section 4.8. A final section with results for the

classical primal–dual affine–scaling direction follows the concluding remarks.

4.2 The centering effect of primal–dual Dikin steps

The ratio

$$\kappa(XS) := \frac{\lambda_{\max}(XS)}{\lambda_{\min}(XS)} = \frac{\lambda_{\max}(V^2)}{\lambda_{\min}(V^2)}$$

was introduced in Section 3.1.3 as a measure of centrality of the pair (X, S). Recall that $\kappa(XS) \geq 1$ for all feasible pairs (X, S) with equality on the central path only.

We saw in Chapter 3 that sufficiently short primal–dual Dikin steps maintain proximity to the central path in terms of κ. In particular, if $\kappa(XS) \leq \tau_0$ holds for the starting solution, it holds for all iterates thereafter.

To gain a better understanding of this centering effect, we turn to a related 'centrality function':

$$\begin{aligned} \Psi(X, S) &:= n \log \frac{\frac{1}{n}\sum_{i=1}^n \lambda_i(XS)}{\left(\prod_{i=1}^n \lambda_i(XS)\right)^{1/n}} \\ &= -\log\det(XS) + n \log \mathbf{Tr}\,(XS) - n \log n. \end{aligned}$$

The function Ψ is determined by the ratio of the arithmetic and geometric means of the eigenvalues of XS. By the arithmetic-geometric mean inequality, Ψ is always nonnegative and zero if and only if the pair (X, S) is centered. The following inequalities show that κ and Ψ are closely related:[1]

$$\log \kappa - 2 \log 2 \leq \Psi(X, S) \leq (n-1) \log \kappa. \tag{4.1}$$

The primal–dual Dikin direction is a descent direction for Ψ, unless (X, S) are perfectly centered. Too see this, we define the block diagonal matrices

$$\Delta W := \begin{bmatrix} \Delta X & 0 \\ 0 & \Delta Z \end{bmatrix}, \quad \nabla\Psi := \begin{bmatrix} \nabla_X \Psi & 0 \\ 0 & \nabla_Z \Psi \end{bmatrix},$$

[1] The inequalities in (4.1) will not be used again and only serve to show that κ is bounded in terms of Ψ. For a proof of (4.1) and extended discussion of the function Ψ the reader is referred to [56], where other bounds are also given.

such that the directional derivative of Ψ along $(\Delta X, \Delta Z)$ is given by

$$\langle \nabla\Psi(X,S), \Delta W\rangle = \mathbf{Tr}\ (\nabla\Psi(X,S)\Delta W)\,.$$

Lemma 4.2.1 *The directional derivative of Ψ along the primal–dual Dikin step direction satisfies*

$$\mathbf{Tr}\ (\nabla\Psi(X,S)\Delta W) \leq 0$$

with equality holding on the central path only.

Proof:

Using the expressions:[2]

$$\nabla_X\Psi(X,S) = -X^{-1} + \frac{n}{\mathbf{Tr}\,(XS)}S, \quad \nabla_S\Psi(X,S) = -S^{-1} + \frac{n}{\mathbf{Tr}\,(XS)}X,$$

it is easy to verify that the directional derivative is given by

$$\begin{aligned}
\mathbf{Tr}\ (\nabla\Psi(X,S)\Delta W) &= \mathbf{Tr}\ (\nabla_X\Psi(X,S)\Delta X + \nabla_S\Psi(X,S)\Delta S) \\
&= \mathbf{Tr}\left((D_X + D_S)\left(\frac{n}{\|V\|^2}V - V^{-1}\right)\right) \\
&= \mathbf{Tr}\left(D_V\left(\frac{n}{\|V\|^2}V - V^{-1}\right)\right).
\end{aligned}$$

Substituting the primal–dual Dikin direction $D_V = \frac{-V^3}{\|V^2\|}$ yields

$$\begin{aligned}
\mathbf{Tr}\ (\nabla\Psi(X,S,\mu)\Delta W) &= \mathbf{Tr}\left(\frac{-nV^4}{\|V\|^2\|V^2\|} + \frac{V^2}{\|V^2\|}\right) \\
&= \frac{\|V\|^2}{\|V^2\|}\left[1 - n\frac{\|V^2\|^2}{\|V\|^4}\right].
\end{aligned}$$

The right hand side expression is always nonpositive by the inequality

$$\|V\|^2 \leq \sqrt{n}\|V^2\| \tag{4.2}$$

(which follows from the Cauchy-Schwartz inequality). Equality holds in (4.2) if and only if $V = \mu I$ for some $\mu > 0$, *i.e.* if and only if X and S are on the central path. □

[2]The required calculus results may be found in Appendix A.

It is therefore clear that the primal–dual Dikin step direction has a centering component (if a sufficiently short step length is used).

Example 4.2.1 *The centering effect is clearly visible in Figure 4.1, for the small example with data*

$$A_1 = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 6 & 4 \\ 3 & 4 & -2 \end{bmatrix},$$

and with feasible starting solution

$$X^0 = \begin{bmatrix} 2.2 & 0.1 & 0.1 \\ 0.1 & 1.5 & 0.08 \\ 0.1 & 0.08 & 1.5 \end{bmatrix}, \; S^0 = \begin{bmatrix} 1.5 & 0.005 & 0 \\ 0.005 & 0.95 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}.$$

The minimum and maximum eigenvalues of XS are plotted at successive iterations for the short step primal–dual Dikin-type method. In this figure the central path corresponds to the diagonal where largest and smallest eigenvalues are equal.

4.3 Complexity analysis in a potential reduction framework

The idea of potential reduction methods is to reduce a potential function by an absolute constant at each iteration. This ensures a polynomial bound on the total iteration count. The so-called Tanabe-Todd-Ye potential function [107, 109] is obtained by adding an additional 'duality gap' term to the 'centrality' function Ψ:

$$\begin{aligned} \phi(X,S) &:= \nu\sqrt{n}\log \mathbf{Tr}\,(XS) + \Psi(X,S) \\ &= (n+\nu\sqrt{n})\log \mathbf{Tr}\,(XS) - \log\det(XS) - n\log n, \end{aligned}$$

where $\nu \geq 1$ determines the relative 'weight' given to the duality gap term. The duality gap $\mathbf{Tr}\,(XS)$ tends to zero as ϕ tends to minus infinity. In particular, one has

$$\mathbf{Tr}\,(XS) \leq \exp\left(\frac{\phi(X,S)}{\nu\sqrt{n}}\right).$$

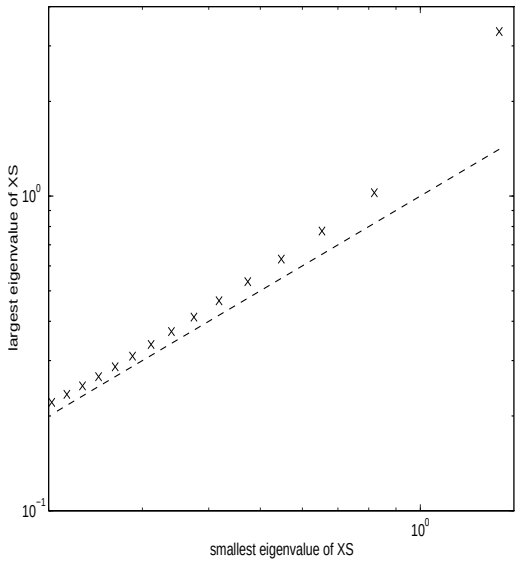

Figure 4.1: The centering effect of primal–dual Dikin steps, seen from iterates of the short step algorithm. The dashed line corresponds to the central path.

If the potential function is decreased by an absolute constant c_{red} (independent of n) at each iteration, then at most

$$\left\lceil \frac{\nu\sqrt{n}L + \Psi(X^0, S^0)}{c_{red}} \right\rceil \tag{4.3}$$

steps are needed to satisfy the convergence condition $\mathbf{Tr}\,(XS) \leq \epsilon$, where (X^0, S^0) are the strictly feasible starting solutions and $L = \log(\mathbf{Tr}\,(X^0S^0)/\epsilon)$, as before.[3]

It was already shown in Lemma 4.2.1 that the primal–dual Dikin step direction is a descent direction for the 'centrality' function Ψ. It is therefore not surprising that it is also a descent direction for ϕ.

Lemma 4.3.1 *The directional derivative of ϕ with respect to the primal–dual Dikin step direction $(\Delta X, \Delta S)$ satisfies*

$$\mathbf{Tr}\,(\nabla\phi(X, S), \Delta W) \leq -\nu.$$

[3]For a proof of this theorem, see *e.g.* [111].

Proof:

Recall that the directional derivative with respect to Ψ is nonpositive, and that

$$\phi(X,S) = \nu\sqrt{n}\log \mathbf{Tr}\,(XS) + \Psi(X,S).$$

We therefore have

$$\begin{aligned}\mathbf{Tr}\,(\nabla\phi(X,S)\Delta W) &\leq \nu\sqrt{n}\mathbf{Tr}\,(\nabla_X\left[\log \mathbf{Tr}\,(XS)\right]\Delta X + \nabla_S\left[\log \mathbf{Tr}\,(XS)\right]\Delta S)\\ &= \frac{\nu\sqrt{n}}{\mathbf{Tr}\,(XS)}\left(\mathbf{Tr}\,(S\Delta X + X\Delta S)\right)\\ &= \frac{\nu\sqrt{n}}{\mathbf{Tr}\,(XS)}\mathbf{Tr}\,(VD_V).\end{aligned}$$

Using $D_V = -V^3/\|V^2\|$ and simplifying yields

$$\begin{aligned}\mathbf{Tr}\,(\nabla\phi(X,S)\Delta W) &\leq -\sqrt{n}\nu\frac{\|V^2\|}{\|V\|^2}\\ &\leq -\nu,\end{aligned}$$

where the second inequality follows from $\|V\|^2 \leq \sqrt{n}\|V^2\|$. □

We now show that a primal–dual Dikin step of length $1/(\tau_0\sqrt{n})$ reduces ϕ by a constant if $\nu = \tau_0\sqrt{n}$. This leads to the same $O(\tau_0 nL)$ worst-case iteration bound as was established in Section 3.4.

To this end, the following lemma first gives a bound for the change of ϕ brought about by a general step αD_V.

Lemma 4.3.2 *A primal–dual step αD_V reduces the potential function ϕ by at least*

$$\Delta\phi \geq -\alpha(n+\nu\sqrt{n})\mathbf{Tr}\,\left(\frac{VD_V}{\|V\|^2}\right) + \alpha\mathbf{Tr}\,\left(V^{-1}D_V\right) - \psi(-\alpha\sqrt{\varepsilon_{NG}}),$$

where

$$\psi(t) := t - \log(1+t), \quad t > -1 \in \mathbb{R}$$

and

$$\varepsilon_{NG} := \left\|X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}\right\|^2 + \left\|Z^{-\frac{1}{2}}\Delta Z Z^{-\frac{1}{2}}\right\|^2. \tag{4.4}$$

Proof:

By the definition of the potential function one has:

$$\begin{aligned}
\Delta\phi &\equiv \phi(X,Z) - \phi(X+\alpha\Delta X, Z+\alpha\Delta Z) \\
&= (n+\nu\sqrt{n})\log\left[\frac{\mathbf{Tr}\,(XZ)}{\mathbf{Tr}\,(XZ)+\alpha\mathbf{Tr}\,(X\Delta Z+Z\Delta X)}\right] \\
&\quad + \log\left[\frac{\det(X+\alpha\Delta X)}{\det X}\right] + \log\left[\frac{\det(Z+\alpha\Delta Z)}{\det Z}\right] \\
&= (n+\nu\sqrt{n})\log\left[\frac{\|V\|^2}{\|V\|^2+\alpha\mathbf{Tr}\,(VD_V)\|}\right] \\
&\quad + \log\Big(\det(X^{-1})\det[X+\alpha\Delta X]\Big) + \log\Big(\det(Z^{-1})\det[Z+\alpha\Delta Z]\Big) \\
&= (n+\nu\sqrt{n})\log\left[\frac{1}{1+\alpha\mathbf{Tr}\,(VD_V)/\|V\|^2}\right] \\
&\quad + \log\det[I+\alpha X^{-1}\Delta X] + \log\det[I+\alpha Z^{-1}\Delta Z].
\end{aligned}$$

To proceed, the following inequality is needed (for a proof, see *e.g.* [98]):

$$\sum_{i=1}^{n}\log(1+x_i) \geq \sum_{i=1}^{n} x_i - \psi(-\|x\|), \quad \forall x\in\mathbb{R}^n,\ \|x\|<1,$$

where

$$\psi(t) := t - \log(1+t), \quad t > -1 \in \mathbb{R} \tag{4.5}$$

(see Figure 4.2). The above inequality can be applied to the eigenvalues of $X^{-1}\Delta X$ and $Z^{-1}\Delta Z$, to obtain the inequality

$$\log\det[I+\alpha X^{-1}\Delta X]+\log\det[I+\alpha Z^{-1}\Delta Z] \geq \mathbf{Tr}\,(X^{-1}\Delta X+Z^{-1}\Delta Z)+\psi(\alpha\sqrt{\varepsilon_{NG}})$$

with ε_{NG} as defined in (4.4).

Thus we have obtained the relation:

$$\begin{aligned}
\Delta\phi &\geq (n+\nu\sqrt{n})\log\left[\frac{1}{1+\alpha\mathbf{Tr}\,(VD_V)/\|V\|^2}\right] + \alpha\mathbf{Tr}\,(X^{-1}\Delta X + Z^{-1}\Delta Z) \\
&\quad -\psi(-\alpha\sqrt{\varepsilon_{NG}}) \\
&= -(n+\nu\sqrt{n})\log\left[1+\alpha\mathbf{Tr}\,(VD_V)/\|V\|^2\right] + \alpha\mathbf{Tr}\,\left(V^{-1}D_V\right) - \psi(-\alpha\sqrt{\varepsilon_{NG}}).
\end{aligned}$$

Using the well-known inequality $-\log(1+x) \geq -x$ completes the proof. □

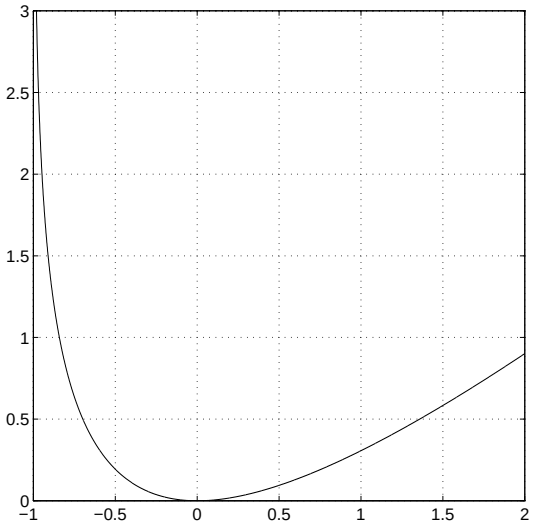

Figure 4.2: Graph of the function $\psi(t) = t - \log(1+t)$

Corollary 4.3.1 *Let $\alpha := \frac{1}{\tau_0\sqrt{n}}$. A primal–dual Dikin step of length α reduces the potential function ϕ by at least*

$$\phi(X,S) - \phi(X + \alpha\Delta X, S + \alpha\Delta S) \geq \frac{\nu}{\tau_0\sqrt{n}} - 0.5208.$$

Proof:

Substituting the Dikin-type step

$$D_V := \frac{-V^3}{\|V\|^2}$$

in Lemma 4.3.2 yields

$$\Delta\phi \geq \nu\alpha - \psi\left(-\alpha\sqrt{\varepsilon_{NG}}\right). \tag{4.6}$$

We can now substitute the step length $\alpha = 1/(\tau_0\sqrt{n})$ in (4.6) and use the bound $\kappa(XS) \leq \varepsilon_{NG}$ from Lemma 3.2.1. Recall from Section 3.4 that this step length maintains the proximity relation $\kappa(XS) \leq \tau_0$ and is always feasible. This substitution yields

$$\Delta\phi \geq \frac{\nu}{\tau_0\sqrt{n}} - \psi\left(\frac{-1}{\sqrt{\tau_0 n}}\right).$$

Since $\tau_0 \geq 1$ and $n \geq 2$ it follows that

$$\psi\left(\frac{-1}{\sqrt{\tau_0 n}}\right) \leq \psi\left(\frac{-1}{\sqrt{2}}\right) < 0.5208.$$

This completes the proof. □

Corollary 4.3.2 *If $\nu = \tau_0\sqrt{n}$ then $\Delta\phi \geq 0.47$.*

The polynomial complexity of the short-step primal–dual Dikin method now follows from (4.3). Formally we have the following theorem.

Theorem 4.3.1 *The primal–dual Dikin short-step algorithm requires at most*

$$\left\lceil \frac{\tau_0 nL + \Psi(X^0, S^0)}{0.47} \right\rceil$$

iterations to compute a strictly feasible pair (X^*, S^*) satisfying **Tr** $(X^*S^*) \leq \epsilon$*, if the initial (strictly feasible) pair (X^0, S^0) satisfies $\kappa(X^0S^0) \leq \tau_0$.*

Remark: The above complexity analysis of the short-step primal–dual Dikin algorithm covers the special case of LP, but has not been published for LP previously. Different complexity analysis for the primal–dual Dikin step method in the LP case is done in [49].

4.4 Longer steps via plane searches of the potential

The short step length which was used in the analysis in the previous section, namely

$$\alpha = \frac{1}{\tau_0\sqrt{n}},$$

guarantees both the feasibility and centrality conditions at each step. In practice it is desirable to take much longer steps, though. Moreover, the performance of the algorithm can further be enhanced by using different step lengths in the primal and dual directions. Once the primal–dual Dikin step direction $(\Delta X, \Delta S)$ has

been computed, step length parameters must be chosen to ensure feasible steps. In particular one must find α, β such that

$$X(\alpha) = X + \alpha \Delta X \succ 0, \quad S(\beta) = S + \beta \Delta S \succ 0.$$

This is usually done by performing a plane search on the potential function ϕ. We briefly review the plane search procedure.[4]

The intervals for feasible step lengths in both the ΔX and ΔS directions are calculated first. This is done by computing the eigenvalues of $X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}}$ and $S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}}$, *i.e.* the generalised eigenvalues of the pairs $(X, \Delta X)$ and $(S, \Delta S)$. One has

$$\begin{aligned}
\alpha_{\min} &= \max_{i=1,\ldots,n} \left\{ \frac{-1}{\lambda_i \left(X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \right)} \ : \ \lambda_i \left(X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \right) > 0 \right\}, \\
\alpha_{\max} &= \min_{i=1,\ldots,n} \left\{ \frac{-1}{\lambda_i \left(X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \right)} \ : \ \lambda_i \left(X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \right) < 0 \right\}, \\
\beta_{\min} &= \max_{i=1,\ldots,n} \left\{ \frac{-1}{\lambda_i \left(S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}} \right)} \ : \ \lambda_i \left(S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}} \right) > 0 \right\}, \\
\beta_{\max} &= \min_{i=1,\ldots,n} \left\{ \frac{-1}{\lambda_i \left(S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}} \right)} \ : \ \lambda_i \left(S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}} \right) < 0 \right\}.
\end{aligned}$$

Once these eigenvalues are known, the plane search reduces to the two dimensional minimization problem: find (α^*, β^*) which minimize

$$\begin{aligned}
f(\alpha, \beta) &:= \phi(X + \alpha \Delta X, S + \beta \Delta S) \\
&= (n + \nu\sqrt{n}) \log(\mathbf{Tr}\,(XS) + \alpha \mathbf{Tr}\,(C \Delta X) - \beta b^T \Delta y) \\
&\quad - \sum_{i=1}^{n} \log \left[1 + \alpha \lambda_i \left(X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \right) \right] - \sum_{i=1}^{n} \log \left[1 + \beta \lambda_i \left(S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}} \right) \right].
\end{aligned}$$

The function f is quasiconvex and has a unique minimizer in the feasible rectangle $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$, $\beta_{\min} \leq \beta \leq \beta_{\max}$. An important observation is that the evaluation of $f(\alpha, \beta)$, $\nabla f(\alpha, \beta)$ and $\nabla^2 f(\alpha, \beta)$ can be done in $O(n)$ operations, which means that the plane search can be done efficiently.

[4] For more details on all aspects of the plane search procedure see [112], and the references therein.

4.5 Improved complexity via centering steps

In the case of the primal–dual Dikin step direction, the step lenghts obtained from a plane search may cause a loss of centrality. In order to guarantee that a potential reduction is always possible, all iterates (X, S) must lie in some cone $\kappa(XS) \leq \tau_0$ around the central path.

The obvious solution is to restore 'sufficient centrality' after the primal–dual Dikin step. This means that some centering steps are needed after each primal–dual Dikin step. This gives the resulting algorithm a 'predictor–corrector' nature.

The centering steps actually improve the worst case iteration complexity, since the primal–dual Dikin steps are no longer restricted to yield iterates inside a centrality cone. This means that the analysis can be done using longer steps. The step length $\alpha = \frac{1}{\tau_0\sqrt{n}}$ was used in Lemma 4.3.2 to derive a bound on the potential reduction $\Delta\phi$. This step length guarantees that $\kappa(XS) \leq \tau_0$ will hold for all iterates. If we are only interested in taking feasible steps without maintaining centrality, this step length can be increased to $\alpha = 1/\sqrt{\kappa(XS)}$, as will now be proved.

Lemma 4.5.1 *Let (X, S) be the current strictly feasible iterates. A primal–dual Dikin step $(X + \alpha\Delta X, S + \alpha\Delta S)$ of length*

$$\alpha = \frac{1}{\sqrt{\kappa(XS)}}$$

is feasible.

Proof:

The key observation for the proof is that primal–dual steps satisfying

$$\varepsilon_{NG} := \|X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}\|^2 + \|S^{-\frac{1}{2}}\Delta S S^{-\frac{1}{2}}\|^2 \leq 1,$$

are feasible (see Section 3.2.3). By definition, primal–dual Dikin steps satisfy

$$\varepsilon_D := \|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|^2 \leq 1,$$

and by Lemma 3.2.1 we have $\varepsilon_{NG} \leq \kappa(XS)\varepsilon_D$. The lemma follows. □

We proceed to bound the reduction of the potential for the step length $\alpha := \frac{1}{2\sqrt{\kappa(XS)}}$.

Lemma 4.5.2 *Let $\alpha := \frac{1}{2\sqrt{\kappa(XS)}}$. A primal–dual Dikin step of length α reduces the potential function ϕ by at least*

$$\Delta\phi := \phi(X,S) - \phi(X + \alpha\Delta X, S + \alpha\Delta S) \geq \frac{\nu}{2\sqrt{\kappa(XS)}} - \psi\left(-\frac{1}{2}\right).$$

Proof:

By the proof of Lemma 4.3.2 (eq. (4.6)) one has

$$\Delta\phi \geq \alpha\nu - \psi\left(-\alpha\sqrt{\kappa(XS)}\right).$$

Substitution of the step length $\alpha := \frac{1}{2\sqrt{\kappa(XS)}}$ completes the proof. □

Corollary 4.5.1 *If $\nu = 2\sqrt{\tau_0}$ and the current iterates satisfy $\kappa(XS) \leq \tau_0$, then $\Delta\phi \geq 0.8$.* □

Note that we no longer require $\nu = O(\sqrt{n})$ in order to guarantee the potential reduction. This implies the improved complexity via (4.3), provided that the centering steps also reduce the potential function

$$\phi(X,S) = 2\sqrt{\tau_0 n}\log \mathbf{Tr}\,(XS) + \Psi(X,S) \tag{4.7}$$

by an absolute constant. It is proved in the next section that the centering steps do indeed give such a potential reduction.

4.6 The centering phase

Assume that (X,S) have resulted from a 'long' primal–dual Dikin step and is 'badly centered', in the sense that $\kappa(XS) > \tau_0$. A centering step is now taken by solving

$$\frac{1}{2}(VD_V + D_V V) = \mu I - V^2 \tag{4.8}$$

where $\mu = \mathbf{Tr}\,(V^2)/n = \mathbf{Tr}\,(XS)/n$. The unique solution is given by

$$D_V = \mu V^{-1} - V. \tag{4.9}$$

In terms of X and S this becomes

$$\Delta X + D\Delta S D = \mu X^{-1} - S. \tag{4.10}$$

The duality gap is constant along this direction, if the same step length is used in the ΔX and ΔS directions:

$$\begin{aligned}
\mathbf{Tr}\,(X + \alpha\Delta X)(S + \alpha\Delta S) &= \mathbf{Tr}\,(V^2 + \alpha V D_V) \\
&= \mathbf{Tr}\,(V^2) + \alpha\mathbf{Tr}\,(\mu I - V^2) \\
&= \mathbf{Tr}\,(V^2) + \alpha\left(\frac{\mathbf{Tr}\,(V^2)}{n}\mathbf{Tr}\,(I) - \mathbf{Tr}\,(V^2)\right) \\
&= \mathbf{Tr}\,(V^2) = \mathbf{Tr}\,(XS).
\end{aligned}$$

Let us investigate the effect of this step on our potential

$$\phi(X, S) = (n + \nu\sqrt{n})\log \mathbf{Tr}\,(XS) - \log\det(XS) - n\log n.$$

Since the duality gap will remain constant we need only consider the effect on the second term, *i.e.*

$$\Delta\phi = \log\det\left((X + \alpha\Delta X)(S + \alpha\Delta S)\right) - \log\det(XS).$$

We show that ϕ can be reduced by an absolute constant along the centering direction if the current iterates lie outside the centrality cone. The reduction will be given in terms of the following centrality measure:

$$\delta := \frac{1}{2}\left\|\sqrt{\mu}V^{-1} - \frac{1}{\sqrt{\mu}}V\right\|,$$

introduced by Jiang in [57] (without the constant $\frac{1}{2}$) and further investigated by De Klerk *et al.* in [26]. Only the case where $\mu = \mathbf{Tr}\,(XS)/n$ is of interest here. In this case δ is a special case of a centrality measure introduced by Nesterov and Todd in [82] for self-scaled conic convex programming.

We need to relate the centrality measures δ and κ. To this end, we will use the following result from Jiang [57].

Lemma 4.6.1 *One has*

$$\max\left\{\rho\left(\sqrt{\mu}V^{-1}\right),\rho\left(\frac{1}{\sqrt{\mu}}V\right)\right\}\leq\delta+\sqrt{1+\delta^2},$$

where ρ denotes the spectral radius. □

The lemma implies that

$$\sqrt{\kappa(XS)}\leq\left(\delta+\sqrt{1+\delta^2}\right)^2=2\delta^2+2\delta\sqrt{1+\delta^2}+1. \tag{4.11}$$

Loosely speaking, if $\kappa(XS)$ is 'large', then so is δ. The following lemma is used to show that the potential can be reduced by a constant if $\kappa(XS)$ is 'large enough'.

Lemma 4.6.2 *The potential ϕ can be decreased along the centering direction (4.9) by at least:*

$$\Delta\phi\geq\psi(\xi)$$

where the function ψ is defined in (4.5), and

$$\xi:=\frac{2\delta}{\delta+\sqrt{1+\delta^2}}.$$

Proof:

From the proof of Lemma 4.3.2 one has

$$\Delta\phi\geq\alpha\mathbf{Tr}\left(V^{-1}D_V\right)+\psi(\alpha\sqrt{\varepsilon_{NG}}) \tag{4.12}$$

where ε_{NG} was defined in (4.4). Substituting the centering direction D_V from (4.9) into (4.12) yields

$$\begin{aligned}\Delta\phi &\geq \alpha\mathbf{Tr}\left(V^{-1}(\mu V^{-1}-V)\right)-\psi(-\alpha\sqrt{\varepsilon_{NG}})\\ &= \alpha\mathbf{Tr}\left(\sqrt{\mu}V^{-1}-\frac{1}{\sqrt{\mu}}V\right)^2-\psi(-\alpha\sqrt{\varepsilon_{NG}})\\ &= \alpha\left\|\sqrt{\mu}V^{-1}-\frac{1}{\sqrt{\mu}}V\right\|^2-\psi(-\alpha\sqrt{\varepsilon_{NG}})=4\alpha\delta^2-\psi(-\alpha\sqrt{\varepsilon_{NG}}).\end{aligned}$$

where we have used $\mu = \mathbf{Tr}\,(V^2)/n$ and the definition of δ. The function

$$f(\alpha) := 4\alpha\delta^2 - \psi(-\alpha\sqrt{\varepsilon_{NG}})$$

has a maximizer

$$\alpha^* = \frac{1}{\sqrt{\varepsilon_{NG}}} - \frac{1}{4\delta^2 + \sqrt{\varepsilon_{NG}}}$$

which guarantees the potential reduction reduction

$$\Delta\phi \geq f(\alpha^*) = (4\delta^2/\sqrt{\varepsilon_{NG}}) - \log(1 + 4\delta^2/\sqrt{\varepsilon_{NG}}) = \psi(4\delta^2/\sqrt{\varepsilon_{NG}}).$$

This inequality will still hold if $4\delta^2/\sqrt{\varepsilon_{NG}}$ is replaced by a smaller value, since ψ is monotonically increasing on the positive real line (see Fig. 4.2). Jiang [57] shows that

$$\frac{\delta^2}{\sqrt{\varepsilon_{NG}}} \geq \frac{2\delta}{\delta + \sqrt{1 + \delta^2}} := \xi.$$

One therefore has that

$$\Delta\phi \geq \psi(\xi),$$

which completes the proof. □

As a corollary we fix a value $\tau_0 = 10$ and show that the centering step always reduces the potential by an absolute constant if $\kappa(XS) \geq \tau_0$.

Corollary 4.6.1 *If* $\kappa(XS) \geq 10$ *then* $\Delta\phi > 0.16$.

Proof:

From eq. (4.11) one has that if $\kappa(XS) \geq 10$ then $\delta > 0.6$. Using this value in the above lemma yields the result. □

4.7 The new potential reduction algorithms

We are now in a position to state the potential reduction variant of the primal–dual Dikin step method. Centering steps are taking while $\kappa(XS) \geq \tau_0$, otherwise

a primal–dual Dikin-type step is used.[5] The resulting algorithm is reminiscent of Mizuno-Todd-Ye [72] predictor–corrector algorithms. The difference is that long steps may be taken here, and more than one centering step may be necessary at each iteration. For this reason we will refer to the algorithms as long step predictor-corrector methods.

[5]Alternatively, primal–dual affine–scaling steps may be used for the 'predictor steps'. The relevant analysis may be found in the Section 4.10.

Generic long step predictor–corrector algorithms

Input
A strictly feasible pair (X^0, S^0);

Parameters
A centering parameter $\tau_0 > 1$;
An accuracy parameter $\epsilon > 0$.
A potential function parameter $\nu > 1$ (default $\nu = 2\sqrt{\tau_0}$);

begin
$X := X^0$; $S := S^0$;
while **Tr** $(XS) > \epsilon$ **do**
if $\kappa(XS) \leq \tau_0$ **do** (*Predictor step*)
Compute ΔX, ΔS from (3.11) and (3.2) (Dikin-type steps)
or from (3.13) and (3.2) (Classical affine–scaling steps);
Find $(\alpha, \beta) =$ argmin $\phi(X + \alpha\Delta X, S + \beta\Delta S)$;
$X := X + \alpha\Delta X$;
$S := S + \beta\Delta S$;

else if $\kappa(XS) > \tau_0$ **do** (*Centering phase*)
Compute ΔX, ΔS from (4.10) and (3.2) ;
Find $\alpha =$ argmin $\phi(X + \alpha\Delta X, S + \alpha\Delta S)$;
$X := X + \alpha\Delta X$;
$S := S + \alpha\Delta S$;

end
end
end.

Each step of this algorithm reduces the potential function ϕ in (4.7) by an absolute constant (at least 0.16) by Corollary 4.5.1 and Corollary 4.6.1. This yields the following worst case iteration bound, by (4.3).

Theorem 4.7.1 *The long step predictor-corrector method using Dikin-type pre-*

dictor steps, requires at most

$$\left\lceil \frac{2\sqrt{\tau_0 n}L + \Psi(X^0, S^0)}{0.16} \right\rceil$$

iterations to compute a strictly feasible pair (X^*, S^*) *satisfying* $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$. □

Loosely speaking, we have convergence in $O(\sqrt{n}L)$ iterations, provided that the initial pair (X^0, S^0) is sufficiently centered.

Example 4.7.1 *An illustration of the algorithm is given in Figure 4.3, for the problem defined as in Example 4.2.1, but with the feasible starting solutions*

$$X^0 = \begin{bmatrix} 1 & 0.1 & 0.3 \\ 0.1 & 3 & 0.8 \\ 0.3 & 0.8 & 5 \end{bmatrix},\ S^0 = \begin{bmatrix} 1.05 & 0.005 & 0 \\ 0.005 & 0.95 & 0 \\ 0 & 0 & 1.1 \end{bmatrix}.$$

The primal–dual Dikin steps are marked as 'predictor steps', and the centering steps as 'corrector steps'. Centering steps are taken if $\kappa(XS) > 10$.

For the problems below, the centering line search is replaced by a plane search of Ψ. This is simply the plane search described in Section 4.4 with $\nu = 0$. This results in faster centering than with equal step lengths. The centering step obtained in this way is accepted if it yields a sufficient decrease in ϕ.

4.8 Computational results

In this section the two new potential reduction methods are tested. In other words, for the first method the predictor step is via the Dikin-type direction, and for the second method the classical primal–dual affine–scaling direction is used. The new methods are compared to the well-known potential reduction method of Nesterov and Todd [85], as implemented by Vandenberghe and Boyd [110]. This method uses the search direction

$$\Delta X + D\Delta S D = \bar{\mu}X^{-1} - S, \tag{4.13}$$

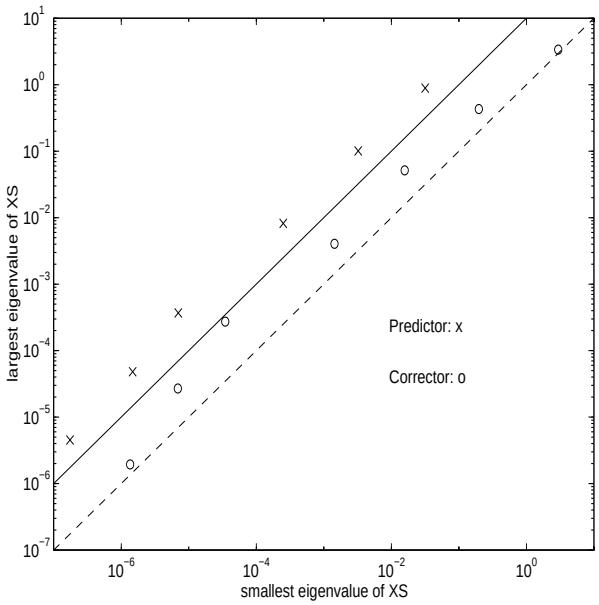

Figure 4.3: Iterates of the primal–dual Dikin potential reduction algorithm. The dashed line corresponds to the central path. The solid line marks the boundary of the centrality cone $\kappa(XS) \leq 10$.

where

$$\bar{\mu} = \frac{\mathbf{Tr}\,(XS)}{n + \nu\sqrt{n}}. \tag{4.14}$$

Nesterov and Todd [85] show that the potential ϕ can always be reduced along this direction by at least $\Delta\phi \geq 0.24$.

Note that the equations (3.11), (3.13) which define the search directions of the new algorithms only differ from (4.13) in terms of the right hand side. The implementations of the new methods could therefore be done by adapting the C-code by Vandenberghe and Boyd [110]. This means that the calculation procedure for the search directions is exactly as described in [110] §5:

1. Perform the generalised eigenvalue decomposition

$$XSL = LV^2$$

where V^2 is diagonal and has the eigenvalues of XS as diagonal elements.

2. Normalize L such that $L^T S L = V$. (One now has a factorization of the scaling matrix, *i.e.* $D = L^T L$.)

3. Find Δy by solving the least squares system

$$\Delta y = \text{argmin}_{y \in \mathbb{R}^m} \left\| rhs + L \left(\sum_{i=1}^{m} y_i A_i \right) L^T \right\|$$

where rhs is determined by the search direction: $rhs = -V$ for classical affine scaling, $rhs = -V^3/|V\|^2$ for the Dikin-step direction, $rhs = \frac{\mathbf{Tr}(V^2)}{n} V^{-1} - V$ for the centering direction, and $rhs = \bar{\mu} V^{-1} - V$ for the algorithm of Nesterov and Todd (see (4.13) and (4.14)).

4. Set $\Delta S = -\sum_{i=1}^{m} \Delta y_i A_i$ and

$$\Delta X = L^T \left(rhs - L \Delta S L^T \right) L.$$

Almost identical amounts of work are therefore done per iteration by the three different methods. Thus only the respective iteration counts (necessary to reduce the initial duality gap by a factor 10^{-8}) are compared below.

The new methods are used with the default settings $\nu = 10$ and $\tau_0 = 10$ except where indicated differently. The algorithm of Nesterov and Todd (NT) employs different settings for ν for the different problem classes as supplied with the software of Vandenberghe and Boyd [110]. These values of ν are therefore 'optimized' to some degree for the test problems considered here. The relevant values are listed in the tables below.

Three well-known classes of test problems are used, namely *educational testing problems*, *matrix norm minimization problems*, and *logarithmic Chebychev approximation*. A more detailed discussion of these problems than given here may be found in [112].

Random matrices (with entries uniformly distributed between zero and one) are used to generate the test problems.

The educational testing problem

This class of problems take the form

$$\max e^T y$$

subject to

$$A - \operatorname{diag}(y) \succeq 0, \quad y \geq 0$$

where e is the all-one vector and $A \succeq 0$. This is a SDP problem with $m = \frac{1}{2}n = \dim(A)$.

In Table 4.1 the average iteration counts of the three methods are compared for five sets of 15 random problems each. The matrix A is generated via $A = \bar{A} + \rho(\bar{A})I$, where $\bar{A}$ is a random symmetric matrix. The algorithm using classical primal–dual affine–scaling predictor steps gives the best results, followed by the method using Dikin-type predictor steps.

(n,m)	(100,50)	(120,60)	(140,70)	(160,80)	(180,90)
NT ($\nu = 100$)	24.1	24.3	24.4	23.5	24.4
P-d. affine sc. ($\nu = 10$)	18.5	18.1	20.1	19.1	19.1
Dikin-type ($\nu = 10$)	19.4	20.1	21.2	21.1	21.6

Table 4.1: Average iteration counts for the Nesterov-Todd (NT) and the new methods on educational testing problems with random data $A = \bar{A} + \rho(\bar{A})I$ where $\bar{A}$ is a random symmetric matrix.

In Table 4.2 the matrix $A \succ 0$ is generated in a different way, namely $A = \bar{A}\bar{A}^T$ where $\bar{A}$ is a random square matrix. The ordering of performance is the same as for Table 4.1 although the difference is smaller. Following Todd *et al.* [108],

(n,m)	(100,50)	(120,60)	(140,70)	(160,80)	(180,90)
NT ($\nu = 100$)	32.9	34.9	42.9	42.2	44.0
P-d. affine sc. ($\nu = 10$)	30.7	32.5	33.2	38.5	39.8
Dikin-type ($\nu = 10$)	33.2	35.4	37.4	40.2	41.6

Table 4.2: Average iteration counts for the Nesterov-Todd (NT) and new methods on educational testing problems with random data $A = \bar{A}\bar{A}^T$ where $\bar{A}$ is a random square matrix.

the experiment is also done with $A = \bar{A}\bar{A}^T$ where $\bar{A}$ is a square Toeplitz matrix

with random first row and column. The results are shown in Table 4.3. For larger problems the new methods are again superior.

(n,m)	(100,50)	(120,60)	(140,70)	(160,80)	(180,90)
NT ($\nu = 100$)	29.5	34.2	42.7	41.8	44.8
P-d. affine sc. ($\nu = 10$)	29.9	32.1	37.6	37.2	39.7
Dikin-type ($\nu = 10$)	32.1	34.7	38.1	40.0	41.9

Table 4.3: Average iteration counts for the Nesterov-Todd (NT) and the new methods on educational testing problems with data $A = \bar{A}\bar{A}^T$ where $\bar{A}$ is a square Toeplitz matrix with random first row and column.

Matrix norm minimization

The goal here is to minimize the 2-norm (maximum eigenvalue norm) of an affine combination of $(p \times q)$ matrices:

$$\min_y \left\{ \|A(y)\|_2 \ : \ A(y) := A_0 + \sum_{i=1}^{k} y_i A_i \right\},$$

which is equivalent to

$$\min t$$

subject to

$$\begin{bmatrix} tI & A(y) \\ A(y)^T & tI \end{bmatrix} \succeq 0$$

which is an SDP problem of dimension $n = p + q$ and $m = k + 1$.

The primal–dual Dikin method is implemented here with $\nu = 1$, *i.e.* less weight is placed on the 'duality gap term' of the potential ϕ. The consequence is that fewer centering steps are taken, *i.e.* mostly primal–dual Dikin steps are taken.

The Nesterov-Todd method performs slightly better on the problems with random data, as seen from Table 4.4. Note however that all three methods require fewer

(n,m)	(100,25)	(120,25)	(140,25)	(160,25)	(180,25)
NT ($\nu = 20$)	11.9	12.5	12.3	13.3	12.9
P-d. affine sc. ($\nu = 10$)	13.1	14.0	13.5	14.2	13.8
Dikin-type ($\nu = 1$)	13.6	14.1	13.3	13.5	13.5

Table 4.4: Average iteration counts for the Nesterov-Todd (NT) and primal–dual Dikin step (Dikin) methods on matrix norm minimization problems with random data.

than 15 iterations for convergence in all cases, and the difference in average iteration count is always less than two iterations. For problems with Toeplitz structured data, the new methods come into their own for the larger problems. For the matrix norm minimization methods there is therefore little to choose between the three methods.

(n,m)	(100,25)	(120,25)	(140,25)	(160,25)	(180,25)
NT ($\nu = 20$)	14.9	15.8	15.6	16.5	18.6
P-d. affine sc. ($\nu = 10$)	18.2	18.7	18.7	17.8	18.9
Dikin-type ($\nu = 1$)	17.6	17.7	18.1	18.2	18.6

Table 4.5: Average iteration counts for the Nesterov-Todd (NT) and the new methods on matrix norm minimization problems with Toeplitz structured data.

Logarithmic Chebychev approximation

The problem is that of approximating the solution of $Ax = b$ if the units of b are on a logarithmic scale. Given data $A = [a_1, \ldots, a_p]^T \in \mathbb{R}^{p \times k}$ and $b \in \mathbb{R}^p$, the problem becomes

$$\min_x \max_{i=1,\ldots,p} \left| \log a_i^T x - \log b_i \right|$$

which is equivalent to

$$\min\left\{t : \frac{1}{t} \leq \frac{a_i^T x}{b_i} \leq t, \quad i = 1, \ldots, p\right\}$$

which in turn is equivalent to

$$\min t$$

subject to

$$\begin{bmatrix} t - \frac{a_i^T x}{b_i} & 0 & 0 \\ 0 & \frac{a_i^T x}{b_i} & 1 \\ 0 & 1 & t \end{bmatrix} \succeq 0, \quad i = 1, \ldots, p,$$

which is an SDP problem of dimension $n = 3p$, $m = k + 1$.

The results are shown for problems with random data in Table 4.6. Here the NT method performs significantly better, requiring four to five fewer iterations on average in most cases. The results for Toeplitz structured data are similar, as seen

(n,m)	(240,121)	(300,151)	(360,181)	(420,211)	(480,241)
NT ($\nu = 20$)	17.3	18.6	18.9	20.5	21.0
P-d. affine sc. ($\nu = 10$)	22.5	22.6	23.6	24.7	25.3
Dikin-type ($\nu = 1$)	21.2	21.7	23.1	23.6	24.4

Table 4.6: Average iteration counts for the Nesterov-Todd (NT) and the new methods on logarithmic Chebychev approximation problems with random data.

from Table 4.7.

4.9 Concluding remarks

Primal–dual affine–scaling methods were analysed in a potential reduction framework. This yielded new proofs of the polynomial worst–case iteration bounds of the short step algorithms, as well as insight into the centering effect of primal–dual Dikin steps. Moreover, the analysis suggested implementable variants of the

(n,m)	(240,121)	(300,151)	(360,181)	(420,211)	(480,241)
NT ($\nu = 20$)	16.9	20.7	21.1	20.7	20.3
P-d. affine sc. ($\nu = 10$)	22.0	22.9	24.0	25.1	24.8
Dikin-type ($\nu = 1$)	20.5	21.9	23.6	24.4	24.4

Table 4.7: Average iteration counts for the Nesterov-Todd (NT) and the new methods on logarithmic Chebychev approximation problems with Toeplitz structured data.

algorithms, which function like 'long–step' predictor–corrector methods. In numerical trails on medium sized problems, the new methods are competitive with the well-known potential reduction method of Nesterov and Todd.

4.10 Results for classical primal–dual affine–scaling

This section contains the results where the primal–dual Dikin step direction $D_V = -V^3/\|V^2\|$ is replaced by the classical primal–dual affine–scaling direction $D_V = -V$ in the above analysis.

Lemma 4.10.1 (Reduction of the potential) *Let (X, S) be the current iterates and $(\Delta X, \Delta S)$ the classical primal–dual affine–scaling directions at (X, S). A step $(X + \alpha\Delta X, S + \alpha\Delta S)$ reduces the potential ϕ by at least*

$$\Delta\phi \geq \nu\sqrt{n}\alpha - \psi\left(\alpha\sqrt{\kappa(XS)}\right).$$

Proof:

Similar to Lemma 4.3.2, but now using $D_V = -V$. □

Corollary 4.10.1 (Complexity of the short step algorithm) *Let $\alpha = 1/(n\tau_0 L)$. The short step algorithm of Section 3.1.4 requires at most*

$$\left\lceil \frac{\tau_0 n L^2 + \Psi(X^0, S^0)}{0.8} \right\rceil$$

iterations to compute a strictly feasible pair (X^*, S^*) *satisfying* $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$, *if the initial (strictly feasible) pair* (X^0, S^0) *satisfies* $\kappa(X^0S^0) \leq \frac{1}{3}\tau_0$.

Proof:

It was shown in Section 3.5 that the short step length $\alpha = 1/(n\tau_0 L)$ guarantees that each step is feasible and all iterates satisfy $\kappa(XS) \leq \tau_0$ if $\kappa(X^0S^0) \leq \frac{1}{3}\tau_0$.[6]

By the last lemma, this step also guarantees a potential reduction of at least

$$\Delta\phi \geq \frac{\nu}{\sqrt{n}\tau_0 L} - \psi\left(-\frac{1}{n\sqrt{\tau_0}L}\right).$$

If we choose $\nu = \sqrt{n}\tau_0 L$, it follows that $\Delta\phi \geq 0.8$. The complexity result now follows from (4.3). □

Lemma 4.10.2 (Condition for a feasible step) *A classical primal–dual affine–scaling step of length*

$$\alpha := \frac{1}{\sqrt{n\kappa(XS)}}$$

is always feasible.

Proof:

By definition, the classical primal–dual affine–scaling direction satisfies

$$\|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|_2 \leq 1.$$

Using $\|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|_2 \leq \sqrt{n}\|V^{-\frac{1}{2}} D_V V^{-\frac{1}{2}}\|$, the proof proceeds as in Lemma 4.5.1. □

Corollary 4.10.2 *A classical primal–dual affine–scaling step of length* $\alpha := \frac{1}{\sqrt{n\kappa(XS)}}$ *reduces the potential* ϕ *by at least* $\Delta\phi \geq 0.47$, *if* $\nu = \sqrt{\tau_0}$ *and* $\kappa(XS) \leq \tau_0$.

[6] Recall that the classical primal–dual affine–scaling steps do not maintain centrality as the primal–dual Dikin steps do. The short step length used here guarantees that convergence is reached before the centrality κ has deteriorated by a factor 3. In other words, if $\kappa(X^0S^0) \leq \frac{1}{3}\tau_0$ then all iterates satisfy $\kappa(XS) \leq \tau_0$.

Proof:

The reduction follows by substituting $\alpha := \frac{1}{\sqrt{n\kappa(XS)}}$ in Lemma 4.10.1. □

The analysis of the centering phase remains unchanged, and the potential reduction method therefore has the following worst–case complexity.

Theorem 4.10.1 (Complexity of the long step predictor-corrector method) *The long step predictor-corrector method using classical primal–dual affine–scaling predictor steps, requires at most*

$$\left\lceil \frac{\sqrt{\tau_0 n}L + \Psi(X^0, S^0)}{0.16} \right\rceil$$

iterations to compute a strictly feasible pair (X^*, S^*) *satisfying* $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$, *where* (X^0, S^0) *are strictly feasible starting solutions and* $L = \log\left(\mathbf{Tr}\,(X^0S^0)/\epsilon\right)$, *as before.* □

Chapter 5

The primal logarithmic barrier method

Primal logarithmic barrier methods for SDP are analysed in this chapter. (These algorithms are also known as primal path-following methods.) In particular, a simple analysis is given of the method of approximate centers of Roos and Vial [99].

5.1 Introduction

In this chapter we extend the so–called method of approximate centers of Roos and Vial [99] from LP to SDP. This is a primal central path following method, and its extension from LP to SDP is surprisingly straightforward. Some of the proofs presented here should be seen as simplification of the proofs first presented by Faybusovich [30, 31], where many of the tools for primal path–following methods were developed. In particular, the centrality measure used in this chapter was introduced and analysed in [30].

The reason for a different presentation of some of the results in [30, 31] is twofold:

1. the analysis presented here is much simpler that that in [30], requiring only basic convex analysis;
2. the self-dual embedding problem of Chapter 2 (Section 2.10.2) for SDP problems in the *symmetric form* can be solved using the primal methods

discussed in this chapter. Thus purely primal methods can be employed in an infeasible-start algorithmic framework.

As before, we denote the unique solution of the system of relaxed optimality conditions:

$$\begin{aligned} \mathbf{Tr}\,(A_i X) &= b_i, \quad X \succ 0, \quad i = 1, \ldots, m \\ \sum_{i=1}^{m} y_i A_i + S &= C, \quad S \succ 0 \\ XS &= \mu I \end{aligned}$$

by $\{X(\mu), y(\mu), S(\mu)\}$. Recall that this solution gives a parametric representation of the central path as a function of μ. The existence and uniqueness of the solution follows from the fact that it corresponds to the unique minimum of the strictly convex primal–dual barrier function[1]

$$f(X, S, \mu) = \frac{1}{\mu}\mathbf{Tr}\,(XS) - \log\det(XS)$$

defined on the primal–dual feasible region $\mathcal{P} \times \mathcal{D}$. The primal–dual barrier f is easily shown to be the sum of the primal and dual barrier functions, defined respectively on $\mathcal{P}$ and $\mathcal{D}$ by

$$f_p(X, \mu) = \frac{1}{\mu}\mathbf{Tr}\,(CX) - \log\det X$$

and

$$f_d(y, S, \mu) = -\frac{1}{\mu}b^T y - \log\det S.$$

The primal central path corresponds to the minimizers $X(\mu)$ of $f_p(X, \mu)$. For this reason μ is referred to as either the *centering parameter* or the *barrier parameter*.

The short step algorithm to be presented follows the primal central path closely, and the search direction ΔX is simply the projected Newton direction of the primal barrier. The algorithm for (P) takes the following form:

[1] To avoid confusion, note that the definitions of the barrier functions given here differs by a constant μ-multiple from what was used in earlier chapters.

Short step primal logarithmic barrier algorithm

Input
A pair (X^0, μ_0) such that X^0 is strictly feasible and sufficiently centered;

Parameters
An accuracy parameter $\epsilon > 0$.
An updating parameter $\theta := \frac{1}{4\sqrt{n}+2}$;
begin
$X := X^0$; $\mu := \mu_0$;
while $n\mu > \epsilon$ **do**
$X := X + \Delta X$;
$\mu := (1 - \theta)\mu$;
end
end.

It will be shown that the above algorithm converges to an ϵ-optimal solution in $O\left(\sqrt{n}\log(1/\epsilon)\right)$ iterations from a sufficiently centered starting point.

Outline of the chapter

The chapter is structured as follows: In Section 5.2 the centrality measure is introduced, which is then related to the primal search direction in Section 5.3. The behaviour of the primal step near the central path is analysed in Section 5.4. The analysis of a centering parameter update in Section 5.5 allows the complexity analysis of Section 5.6. The dual algorithm is briefly discussed in Section 5.7. The chapter ends with a bibliographical note about the long–step version of the algorithm.

The '**vec**' notation will again be used in this chapter. Recall that **vec**(A) is the n^2 dimensional vector obtained by stacking the n columns of A on top of one another, and that **vec**$(A)^T$**vec**$(B) =$ **Tr** $\left(A^T B\right) = \langle A, B\rangle$.

5.2 Measure of centrality

For primal feasible X and a given parameter $\mu > 0$, we define

$$(S(X,\mu), y(X,\mu)) := \arg \min_{S\in\mathcal{S}_n, y\in\mathbb{R}^m} \left\{ \left\| \frac{1}{\mu} X^{\frac{1}{2}} S X^{\frac{1}{2}} - I \right\| \ : \ \sum_{i=1}^{m} y_i A_i + S = C \right\}. \tag{5.1}$$

In other words, $S(X,\mu)$ satisfies the dual feasibility constraints with the semi–definiteness condition relaxed and minimizes the deviation of the pair $(X, S(X,\mu))$ from the central path, where the deviation is quantified using the measure

$$\delta(X,\mu) := \left\| \frac{X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}}}{\mu} - I \right\|.$$

As discussed in Chapter 1, an LP problem is obtained from (P) (resp. (D)) by restricting X (resp. S) to be diagonal. The centrality measure $\delta(X,\mu)$ is precisely the Roos-Vial distance measure in the LP case [99] if X and $S(X,\mu)$ are restricted to be diagonal, as expected.

Note that one has

$$\delta(X,\mu) = 0 \quad \Longleftrightarrow \quad X = X(\mu).$$

We will refer to X as being sufficiently centered if $\delta(X,\mu)$ is smaller than some prescribed tolerance.

The matrix $S(X,\mu)$ plays an important role in the following analysis. In particular, the search direction of the algorithm can be expressed in terms of it, as is shown in the next section.

5.3 The primal step

The projected Newton direction for the primal barrier

$$f_p(X,\mu) = \frac{1}{\mu}\mathbf{Tr}\,(CX) - \log\det X$$

at a given pair (X,μ) is defined as (see [35]):

$$\Delta X = \arg\min_{\Delta X} \langle \nabla f_p(X,\mu), \Delta X\rangle + \frac{1}{2}\left\langle \nabla^2 f_p(X,\mu)\Delta X, \Delta X\right\rangle \tag{5.2}$$

$$\equiv \arg\min_{\Delta X} \mathbf{Tr}\ (\nabla f_p(X,\mu)\Delta X) + \frac{1}{2}\mathbf{Tr}\ \left(\nabla^2 f_p(X,\mu)\Delta X^2\right) \tag{5.3}$$

subject to the feasibility conditions

$$\mathbf{Tr}\,(A_i \Delta X) = 0, \quad i = 1, \ldots, m,$$

where ∇f_p and $\nabla^2 f_p$ denote the gradient and Hessian of f_p respectively. In other words, the projected Newton step minimizes the quadratic Taylor approximation to f_p subject to feasibility of the step direction.

As in the LP case, an explicit expression for ΔX may be obtained. To this end, it is shown in Appendix A that for symmetric X,

$$\nabla f_p(X, \mu) = \frac{1}{\mu} C - X^{-1},$$

and $\nabla^2 f_p(X, \mu) : \mathcal{S}_n \mapsto \mathcal{S}_n$ is the linear operator which satisfies

$$\nabla^2 f_p(X, \mu) \Delta X = X^{-1} \Delta X X^{-1} \quad \forall\, \Delta X \in \mathcal{S}_n.$$

Substitution of the gradient and Hessian in (5.2) yields

$$\Delta X \;=\; \text{argmin}_{\Delta X} \left\{ \mathbf{Tr}\,\left(\frac{C \Delta X}{\mu} \right) - \mathbf{Tr}\,\left(X^{-1} \Delta X \right) + \frac{1}{2} \mathbf{Tr}\,\left(\left(X^{-1} \Delta X \right)^2 \right) \right\},$$

subject to

$$\mathbf{Tr}\,(A_i \Delta X) = 0, \quad i = 1, \ldots, m.$$

The first order optimality conditions for this problem are

$$\begin{aligned} \frac{1}{\mu} C - X^{-1} + X^{-1} \Delta X X^{-1} + \sum_{i=1}^{m} y_i A_i &= 0 \\ \mathbf{Tr}\,(A_i \Delta X) &= 0, \quad i = 1, \ldots, m. \end{aligned}$$

Straightforward manipulation of the optimality conditions yields

$$\mathbf{vec}\left(X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \right) = -\left[I - \mathcal{A}_X{}^T (\mathcal{A}_X \mathcal{A}_X{}^T)^{-1} \mathcal{A}_X \right] \left(\mathbf{vec}\left(\frac{1}{\mu} X^{\frac{1}{2}} C X^{\frac{1}{2}} - I \right) \right), \tag{5.4}$$

where $\mathcal{A}_X$ is the $m \times n^2$ matrix with row vectors $\left[\mathbf{vec}\left(X^{\frac{1}{2}} A_j X^{\frac{1}{2}} \right) \right]$, for $j = 1, \ldots, m$.

Expression (5.4) is simply the orthogonal projection of the vector $\mathbf{vec}(\frac{1}{\mu} X^{\frac{1}{2}} C X^{\frac{1}{2}} - I)$ onto the nullspace of $\mathcal{A}_X$. Note that the row space of $\mathcal{A}_X$ is given by

$$\text{span}\left\{ \mathbf{vec}\left(X^{\frac{1}{2}} A_1 X^{\frac{1}{2}} \right), \ldots, \mathbf{vec}\left(X^{\frac{1}{2}} A_m X^{\frac{1}{2}} \right) \right\}$$

and the nullspace is the orthogonal complement of this space.

Reverting to the space of symmetric matrices $\mathcal{S}_n$, it is clear that the search direction ΔX is obtained via a projection of the matrix $[\frac{1}{\mu}X^{\frac{1}{2}}CX^{\frac{1}{2}} - I]$ onto the orthogonal complement of

$$\text{span}\,\{X^{\frac{1}{2}}A_1X^{\frac{1}{2}}, \ldots, X^{\frac{1}{2}}A_mX^{\frac{1}{2}}\}.$$

The relevant projection[2] operator $P_{\mathcal{A}_X} : \mathcal{S}_n \mapsto \mathcal{S}_n$ is given by

$$P_{\mathcal{A}_X}(M) := \arg\min_{W \in \mathcal{S}_n} \{\|W - M\| \ : \ \mathbf{Tr}\,(X^{\frac{1}{2}}A_iX^{\frac{1}{2}}W) = 0, \ \ i = 1, \ldots, m\}. \tag{5.5}$$

We can now write the search direction ΔX in terms of $S(X, \mu)$.

Lemma 5.3.1 *The primal search direction has the following two representations:*

$$\Delta X = -X^{\frac{1}{2}}\left(P_{\mathcal{A}_X}\left(\frac{X^{\frac{1}{2}}CX^{\frac{1}{2}}}{\mu} - I\right)\right)X^{\frac{1}{2}} = -\left(\frac{XS(X,\mu)X}{\mu} - X\right).$$

Proof:

Note that the first-order optimality conditions for

$$\min_{W \in \mathcal{S}_n}\left\{\left\|W - (\frac{X^{\frac{1}{2}}CX^{\frac{1}{2}}}{\mu} - I)\right\| \ : \ \mathbf{Tr}\,\left(X^{\frac{1}{2}}A_iX^{\frac{1}{2}}W\right) = 0, \ \ i = 1, \ldots, m\right\} \tag{5.6}$$

are

$$\left.\begin{array}{rcl} W - \left(\frac{X^{\frac{1}{2}}CX^{\frac{1}{2}}}{\mu} - I\right) + \sum_{i=1}^m \xi_i X^{\frac{1}{2}}A_iX^{\frac{1}{2}} & = & 0, \\ \mathbf{Tr}\,(A_iX^{\frac{1}{2}}WX^{\frac{1}{2}}) & = & 0, \ \ i = 1, \ldots, m. \end{array}\right\} \tag{5.7}$$

The first–order optimality conditions for

$$\min_{y \in \mathbb{R}^m, S \in \mathcal{S}_n}\left\{\left\|\frac{X^{\frac{1}{2}}SX^{\frac{1}{2}}}{\mu} - I\right\| \ : \ \sum_{i=1}^m y_iA_i + S = C\right\} \tag{5.8}$$

[2]The presentation here is an extension of the LP presentation in [99]. A different but equivalent approach may be found in Nemirovskii and Gahinet [81], where they consider the projection onto span $\{A_1, \ldots, A_m\}^\perp$ relative to the metric induced by the inner product $\langle A, B\rangle_{X^{\frac{1}{2}}} := \mathbf{Tr}\,(AX^{\frac{1}{2}}BX^{\frac{1}{2}})$ for symmetric matrices A and B.

can be written as

$$\left.\begin{array}{rcl} \frac{XSX}{\mu^2} - Q & = & \frac{X}{\mu}, \\ \mathbf{Tr}\,(A_i Q) & = & 0, \quad i = 1, \ldots, m, \\ \sum_{i=1}^m y_i A_i + S & = & C, \end{array}\right\} \tag{5.9}$$

where $Q \in \mathcal{S}_n$. If we denote the solution of System (5.9) by $(y(X,\mu), S(X,\mu), Q(X,\mu))$, it follows that

$$\xi(X,\mu) := \frac{1}{\mu} y(X,\mu) \text{ and } W(X,\mu) := \mu X^{-\frac{1}{2}} Q(X,\mu) X^{-\frac{1}{2}}$$

satisfy the first equation of System (5.7). The second equation of System (5.9) shows that

$$\mathbf{Tr}\,(A_i X^{\frac{1}{2}} W(X,\mu) X^{\frac{1}{2}}) = 0, \quad i = 1, \ldots, m.$$

Thus an optimal solution to problem (5.6) can be constructed from an optimal solution to (5.8). Since both these problems simply involve orthogonal projections onto convex sets, and have unique solutions, the equivalence of the two definitions of ΔX follows. □

Lemma 5.3.1 shows that the search direction ΔX may be split into two terms, say

$$\Delta X := \frac{1}{\mu} \Delta X^a + \Delta X^c,$$

where

$$\Delta X^a := -X^{\frac{1}{2}} \left(P_{\mathcal{A}_X} \left(X^{\frac{1}{2}} C X^{\frac{1}{2}} \right) \right) X^{\frac{1}{2}} \tag{5.10}$$

and

$$\Delta X^c := X^{\frac{1}{2}} \left(P_{\mathcal{A}_X} (I) \right) X^{\frac{1}{2}}. \tag{5.11}$$

The terms ΔX^a and ΔX^c are respectively called the *affine–scaling* and *centering* components of the search direction. We encountered the primal affine–scaling direction in Section 3.6. (It is easy to see that definition (5.10) is equivalent to the definition (3.25) in Section 3.6.) It was mentioned in Section 3.6 that affine–scaling steps alone do not always converge to the optimal solution for SDP problems; it is necessary to add the centering component ΔX^c.

Computation of the search direction in practice

The optimality conditions (5.9) may be solved by rewriting them as

$$\sum_{i=1}^{m} y_i \mathbf{Tr}\left(XA_iXA_j\right) = \mathbf{Tr}\left(XA_jXC\right) - \mu\mathbf{Tr}\left(A_jX\right), \quad j = 1, \ldots, m. \tag{5.12}$$

The solution of this $m \times m$ linear system yields $y(X, \mu)$. The coefficient matrix $[\mathbf{Tr}\left(XA_iXA_j\right)]$ of the linear system (5.12) is symmetric positive definite because the matrices A_i $(i = 1, \ldots, m)$ are linearly independent (see Appendix A, Lemma A.2.2). Letting $S(X, \mu) = \sum_{i=1}^{m} y_i(X, \mu)A_i - C$, the search direction is calculated from

$$\Delta X = -\frac{1}{\mu}XS(X, \mu)X + X.$$

An alternative to the above is to rewrite the definition of $y(X, \mu)$ as the solution of the least squares problem:

$$y(X, \mu) = \text{argmin}_{y \in \mathbb{R}^m} \left\| {\mathcal{A}_X}^T y - \mathbf{vec}\left(I - \frac{1}{\mu}X^{\frac{1}{2}}CX^{\frac{1}{2}}\right)\right\|,$$

after which $S(X, \mu)$ and subsequently ΔX are obtained as before. This least squares solution is in line with the approach in [112] and more recently by Todd *et al.* in [108].

5.4 Behaviour near the central path

Consider a primal update

$$X^+ := X + \Delta X = 2X - \frac{1}{\mu}XS(X, \mu)X.$$

The pair $(X^+, S(X, \mu))$ now satisfies the primal and dual equality constraints with the semidefiniteness requirements relaxed. The next two lemmas show that the semidefiniteness requirements are also satisfied if X is sufficiently centered.

Lemma 5.4.1 *If $X \succ 0$ and $\delta(X, \mu) < 1$, then $S(X, \mu) \succ 0$.*

Proof:

By definition

$$\begin{aligned}\delta(X,\mu)^2 &= \left\| \frac{X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}}}{\mu} - I \right\|^2 \\ &= \mathbf{Tr}\left(\left(\frac{1}{\mu} X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}} - I \right)^2 \right) \\ &= \sum_{i=1}^{n} \left(\frac{1}{\mu} \lambda_i \left(X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}} \right) - 1 \right)^2 .\end{aligned}$$

Using $\delta(X,\mu) < 1$, we have

$$\sum_{i=1}^{n} \left(\frac{1}{\mu} \lambda_i \left(X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}} \right) - 1 \right)^2 < 1,$$

which shows that $\lambda_i \left(X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}} \right) > 0$ $(i = 1 \ldots, n)$, and thus $S(X,\mu) \succ 0$. □

The next step is to show that $X^+ := X + \Delta X$ is feasible if X is sufficiently centered.

Lemma 5.4.2 *Let* $X^+ = X + \Delta X = 2X - \frac{1}{\mu} X S(X,\mu) X$. *If* $X \succ 0$ *and* $\delta(X,\mu) < 1$, *then* $X^+ \succ 0$.

Proof:

Note that X^+ may be written as

$$X^+ = X^{\frac{1}{2}} \left(2I - X^{\frac{1}{2}} \frac{S(X,\mu)}{\mu} X^{\frac{1}{2}} \right) X^{\frac{1}{2}}. \tag{5.13}$$

Because $\delta(X,\mu) < 1$, *i.e.* $\left\| \frac{1}{\mu} X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}} - I \right\| < 1$, it follows that

$$\sum_{i=1}^{n} \lambda_i^2 \left(\frac{X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}}}{\mu} - I \right) < 1.$$

Thus we have

$$\lambda_i \left(\frac{1}{\mu} X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}} \right) \in (0,2), \quad i = 1, \ldots, n$$

which implies

$$\lambda_i \left(2I - X^{\frac{1}{2}} \frac{S(X,\mu)}{\mu} X^{\frac{1}{2}} \right) \in (0,2), \quad i = 1, \ldots, n$$

and consequently $X^+ \succ 0$, by (5.13). □

One also has quadratic convergence of the primal iterate to the central path.

Lemma 5.4.3 *If $X \succ 0$ and $\delta(X,\mu) < 1$ then a primal update $X^+ = 2X - \frac{1}{\mu} X S(X,\mu) X$ satisfies $\delta(X^+,\mu) \leq \delta^2(X,\mu)$.*

Proof:

By definition

$$\begin{aligned}
\delta(X^+,\mu)^2 &= \left\| \frac{X^{+\frac{1}{2}} S(X^+,\mu) X^{+\frac{1}{2}}}{\mu} - I \right\|^2 \\
&\leq \left\| \frac{X^{+\frac{1}{2}} S(X,\mu) X^{+\frac{1}{2}}}{\mu} - I \right\|^2 \\
&= \mathbf{Tr} \left(\left(\frac{1}{\mu} S(X,\mu) X^+ - I \right)^2 \right).
\end{aligned}$$

Substituting $X^+ = 2X - \frac{1}{\mu} X S(X,\mu) X$ yields

$$\begin{aligned}
\delta(X^+,\mu)^2 &\leq \mathbf{Tr} \left(\frac{1}{\mu} S(X,\mu) \left[2X - \frac{1}{\mu} X S(X,\mu) X \right] - I \right)^2 \\
&= \mathbf{Tr} \left(\frac{1}{\mu} S(X,\mu) X - I \right)^4 \\
&\leq \left(\mathbf{Tr} \left(\frac{1}{\mu} S(X,\mu) X - I \right)^2 \right)^2 \\
&= \left\| \frac{1}{\mu} X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}} - I \right\|^4 = \delta^4(X,\mu),
\end{aligned}$$

where the last inequality follows from the Cauchy–Schwartz inequality. □

This last result was first established by Faybusovich [31], and later in [43] and [10]. It was also obtained in the general setting of convex programming problems in conic form for self-scaled cones by Nesterov and Todd [85].

5.5 Updating the centering parameter

Once the primal iterate X is sufficiently centered, *i.e.* $\delta(X,\mu) \leq \tau$ for some tolerance τ, the parameter μ can be reduced. To fix our ideas, we update the barrier parameter in such a way that we will still have $\delta(X,\mu^+) \leq \frac{1}{2}$ after an update $\mu \rightarrow \mu^+$. The following Newton step will then yield a feasible X^+ satisfying $\delta(X^+,\mu^+) \leq \frac{1}{4}$, by Lemma 5.4.3.

In order to realize these ideas, the effect of a μ-update on the proximity measure must be analysed.

Lemma 5.5.1 *Define a μ–update by $\mu^+ := (1-\theta)\mu$, with $0 < \theta < 1$. It then follows that*

$$\delta(X,\mu^+) \leq \frac{1}{1-\theta}(\delta(X,\mu) + \theta\sqrt{n}).$$

Proof:

Using the definition of $S(X,\mu^+)$ we may write

$$\begin{aligned}
\delta(X,\mu^+) &= \left\| \frac{X^{\frac{1}{2}} S(X^+,\mu^+) X^{\frac{1}{2}}}{(1-\theta)\mu} - I \right\| \\
&\leq \left\| \frac{X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}}}{(1-\theta)\mu} - I \right\| \\
&\leq \frac{1}{1-\theta} \left(\left\| \frac{X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}}}{\mu} - I \right\| + \theta \left\| I \right\| \right) \\
&= \frac{1}{1-\theta} \left(\delta(X,\mu) + \theta\sqrt{n} \right),
\end{aligned}$$

where the second inequality follows from the triangle inequality. □

The above result enables us to choose an updating parameter θ which guarantees that the primal iterate remains sufficiently centered with respect to the new parameter $\mu^+ := (1-\theta)\mu$.

Lemma 5.5.2 *Let $\delta(X,\mu) \leq \frac{1}{2}$ and $\theta = 1/(4\sqrt{n}+2)$. After a step $X^+ = X+\Delta X$ and a subsequent update $\mu^+ = (1-\theta)\mu$, one has $\delta(X^+,\mu^+) \leq \frac{1}{2}$.*

Proof:

Using Lemma 5.5.1 and Lemma 5.4.3 successively we get

$$\begin{aligned}\delta(X^+,\mu^+) &\leq \frac{1}{1-\theta}\left(\delta(X^+,\mu)+\theta\sqrt{n}\right)\\ &\leq \frac{1}{1-\theta}\left(\delta^2(X,\mu)+\theta\sqrt{n}\right).\end{aligned}$$

Substitution of $\theta = 1/(4\sqrt{n}+2)$ gives

$$\delta(X^+,\mu^+) \leq \frac{4\sqrt{n}+2}{4\sqrt{n}+1}\left(\frac{1}{4}+\frac{\sqrt{n}}{4\sqrt{n}+2}\right) = \frac{1}{2},$$

which completes the proof. □

Dynamic μ-updates

It is easily verified that if $\delta(X,\mu) \leq \frac{1}{2}$ the dynamic update

$$\theta = \frac{\frac{1}{2}-\delta(X,\mu)}{\sqrt{n}+\frac{1}{2}} \geq \frac{1}{4\sqrt{n}+2}$$

ensures that $\delta(X,\mu^+) \leq \frac{1}{2}$, if $\mu^+ = (1-\theta)\mu$. A natural question is whether it is possible to find the smallest value of μ^+ such that the proximity condition $\delta(X,\mu^+) \leq \frac{1}{2}$ still holds. This is indeed possible; the key observation is that $\delta(X,\mu)$ can be rewritten as

$$\delta(X,\mu) = \left\| X^{-\frac{1}{2}}\left(\Delta X^c + \frac{1}{\mu}\Delta X^a\right)X^{-\frac{1}{2}}\right\|,$$

by the definition of δ and Lemma 5.3.1.

Denoting $D^a := X^{-\frac{1}{2}}\Delta X^a X^{-\frac{1}{2}}$ and $D^c := X^{-\frac{1}{2}}\Delta X^c X^{-\frac{1}{2}}$, we see that the smallest value of μ^+ which still satisfies $\delta(X,\mu^+) \leq \frac{1}{2}$ is the smallest positive root of the equation

$$\delta(X,\mu) = \left\| D^c + \frac{1}{\mu} D^a \right\| = \frac{1}{2}.$$

Squaring both sides of the last equation yields the following quadratic equation in $\frac{1}{\mu}$:

$$\frac{1}{\mu^2}\left\|D^a\right\|^2 + \frac{2}{\mu}\mathbf{Tr}\,(D^a D^c) + \left\|D^c\right\|^2 - \frac{1}{4} = 0,$$

which can be solved to obtain the desired value μ^+.[3]

5.6 Complexity analysis

To prove the polynomial complexity of the algorithm, we need the following lemma which bounds the duality gap in terms of the centrality measure δ.

Lemma 5.6.1 *If $\delta(X,\mu) \leq 1$ then*

$$\mu\left(n - \delta(X,\mu)\sqrt{n}\right) \leq \mathbf{Tr}\,(CX) - b^T y(X,\mu) \leq \mu\left(n + \delta(X,\mu)\sqrt{n}\right).$$

Proof:

Note that

$$\mathbf{Tr}\,(CX) - b^T y(X,\mu) = \mathbf{Tr}\,(XS(X,\mu)) = \mathbf{Tr}\,(X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}}).$$

Using the Cauchy-Schwartz inequality yields

$$\delta(X,\mu)\sqrt{n} = \left\| \frac{X^{\frac{1}{2}} S(X,\mu) X^{\frac{1}{2}}}{\mu} - I \right\| \|I\| \geq \left| \frac{\mathbf{Tr}\,(XS(X,\mu))}{\mu} - n \right|,$$

which implies that

$$n - \delta(X,\mu)\sqrt{n} \leq \frac{\mathbf{Tr}\,(XS(X,\mu))}{\mu} \leq n + \delta(X,\mu)\sqrt{n},$$

[3]This dynamic updating strategy is the extension of the strategy for LP described in [98], §6.8.3.

which in turn gives the required result. □

We can now derive the worst case complexity bound of the algorithm.

Theorem 5.6.1 *Let $\epsilon > 0$ be an accuracy parameter, $\theta = \frac{1}{4\sqrt{n}+2}$ and $\mu^0 > 0$. Let $X^0 \succ 0$ be a strictly feasible starting point such that $\delta(X^0, \mu^0) \leq \frac{1}{2}$. The Algorithm terminates after at most $\left\lceil 6\sqrt{n} \log \frac{n\mu^0}{\epsilon} \right\rceil$ steps, the last generated points X and $S(X, \mu)$ are strictly feasible, and the duality gap is bounded by* $\mathbf{Tr}\,(XS(X, \mu)) \leq \frac{3}{2}\epsilon$.

Proof:

After each iteration of the algorithm X will be strictly feasible, and $\delta(X, \mu) \leq 1/2$, due to Lemma 5.5.2. After the k-th iteration one has $\mu = (1 - \theta)^k \mu_0$. The algorithm stops if k is such that

$$n\mu_0(1 - \theta)^k < \epsilon.$$

Taking logarithms on both sides, this inequality reduces to

$$-k \ln(1 - \theta) > \log \frac{n\mu^0}{\epsilon}.$$

Since $-\ln(1 - \theta) > \theta$, this will certainly hold if

$$k\theta > \log \frac{n\mu^0}{\epsilon},$$

which implies

$$k > 6\sqrt{n} \log \frac{n\mu^0}{\epsilon}$$

for the default setting $\theta := \frac{1}{4\sqrt{n}+2}$. This proves the first statement in the theorem. Now let X be the last generated point, then it follows from Lemma 5.4.1 that $S(X, \mu) \succ 0$. Moreover, the duality gap is then bounded by

$$\begin{aligned} \mathbf{Tr}\,(XS(X, \mu)) &\leq n\mu \left(1 + \frac{\delta(X, \mu)}{\sqrt{n}}\right) \\ &\leq \epsilon \left(1 + \frac{\delta(X, \mu)}{\sqrt{n}}\right) \leq \frac{3}{2}\epsilon. \end{aligned}$$

where the first inequality follows from Lemma 5.6.1. This completes the proof. □

5.7 The dual algorithm

The algorithm for the dual problem is perfectly analogous to that of the primal problem. If one defines

$$X(S,\mu) := \arg\min_{X\in\mathcal{S}} \left\{ \left\| \frac{S^{\frac{1}{2}} X S^{\frac{1}{2}}}{\mu} - I \right\| \;:\; \mathbf{Tr}\, A_i X = b_i, \quad i = 1,\ldots,m \right\}$$

for a strictly feasible dual variable $S \succ 0$, then the first–order optimality conditions which yield $X(S,\mu)$ are

$$\begin{aligned} \frac{S}{\mu}\left[\frac{XS}{\mu} - I\right] + \sum_{i=1}^{m} y_i A_i &= 0 \\ \mathbf{Tr}\,(A_i X) &= b_i, \quad i = 1,\ldots,m. \end{aligned}$$

If we now define

$$\delta(S,\mu) := \left\| \frac{S^{\frac{1}{2}} X(S,\mu) S^{\frac{1}{2}}}{\mu} - I \right\|,$$

then we can repeat the analysis for the primal algorithm, but with the roles of X and S interchanged. The search direction of the algorithm, *i.e.* the projected Newton direction of the dual barrier f_d, becomes

$$\Delta S = S^{\frac{1}{2}} \left(I - \frac{1}{\mu} S^{\frac{1}{2}} X(S,\mu) S^{\frac{1}{2}} \right) S^{\frac{1}{2}}$$

which uniquely determines Δy, via $\sum_{i=1}^{m} \Delta y_i A_i = -\Delta S$.

This results in the following algorithm:

Short step dual logarithmic barrier algorithm

Input
A strictly dual feasible pair (S^0, y^0);
A parameter $\mu_0 > 0$ such that $\delta(S^0, \mu_0) \leq \frac{1}{2}$.
Parameters
An accuracy parameter $\epsilon > 0$.
An updating parameter $\theta := \frac{1}{4\sqrt{n}+2}$;
begin
$S := S^0$; $\mu := \mu_0$;
while $n\mu > \epsilon$ **do**
$S := 2S - \frac{1}{\mu} SX(S, \mu)S$;
$\mu := (1 - \theta)\mu$;
end
end

Due to the symmetry in the analysis, the dual algorithm has the same complexity bound as the primal algorithm.

5.8 Bibliographical note: the long step method

The primal method presented here has also been extended to use larger μ-updates by Faybusovich [31] and later by Anstreicher and Fampa [10].

Given the results in this chapter, this 'large update' (or 'long step') algorithm can be derived as a mechanical extension of the corresponding LP analysis by Roos *et al.* [98], §6.9.

The algorithm in question performs damped Newton steps with respect to a given value of μ until $\delta(S, \mu) \leq \frac{1}{\sqrt{2}}$ holds for the last iterate S (*inner iterations*). Only then is μ reduced by a fixed fraction via $\mu \leftarrow (1 - \theta)\mu$ (*outer iteration*). The algorithm can be stated as follows:

Long step dual logarithmic barrier method

Input
A strictly feasible starting pair (S^0, y^0);

Parameters
A centering parameter $\tau_0 > 1$ (default $\tau_0 = \frac{1}{\sqrt{2}}$);
An accuracy parameter $\epsilon > 0$.
An updating parameter $\theta < 1$.

begin
$S := S^0$; $y := y^0$;
while **Tr** $(XS) > \epsilon$ **do**
if $\delta(S, \mu) \leq \tau_0$ **do** (*outer iteration*)
$\mu := (1 - \theta)\mu$;
else if $\delta(S, \mu) > \tau_0$ **do** (*inner iteration*)
Compute $(\Delta S, \Delta y)$;
Find $\alpha =$ argmin $f_d(y + \alpha\Delta y, S + \alpha\Delta S, \mu)$;
$S := S + \alpha\Delta S$;
$y := y + \alpha\Delta y$;

end
end
end

This algorithm has the same worst-case iteration bound as in the LP case: if $\theta = O(1)$ the iteration bound is $O(nL)$ and if $\theta = O\left(\frac{1}{\sqrt{n}}\right)$ the iteration bound becomes $O(\sqrt{n}L)$.

Chapter 6

Primal–dual logarithmic barrier methods

In this chapter primal–dual path following methods are described. The analysis is done for the Nesterov-Todd search direction, and using the centrality measure of Jiang [57] (the norm of the directional derivative of the primal–dual log-barrier function). For this centrality measure, a weaker condition for a feasible full Newton step than previously known is established, and quadratic convergence to target points on the central path is shown for the first time. Moreover, it is shown how to compute large dynamic barrier parameter updates which still allow full Newton steps.

6.1 Introduction

Primal-dual path following methods have emerged as the most successful interior point algorithms for linear programming (LP). Predictor-corrector methods are particularly in favour, following successful implementations [71].

The extension of algorithms from linear to semidefinite programming (SDP) has followed the same trends. Recent attention has focused on path following algorithms. Primal path following algorithms are studied in [30] (general analysis), [10] (long step method), and [43] (full Newton step method).

Primal-dual path following methods employ different search directions which

arise from different linearization strategies of the relaxed optimality conditions. A comparison of the best known search directions can be found in [4] and [108].

One of the popular primal-dual directions is the so-called Nesterov-Todd (NT) direction, introduced in [85]. (Recently Kojima *et al.* [62] showed the NT direction to be a special case of the primal–dual directions for monotone semidefinite complementarity problems introduced in [65].)

A long step primal-dual path following method using the NT direction was recently presented by Jiang [57]; in his paper an extension of the LP analysis by Jansen *et al.* [51] is presented. This method targets a specific point on the primal-dual central path, which is then updated if the current iterates are 'close enough' to it. The novelty of this analysis lies in the use of a new centrality measure, which is analogous to the LP measure in [51].

In this chapter we extend and refine this analysis – a weaker sufficient condition for full Newton steps is derived, and quadratic convergence to target points on the central path is proved. Moreover, a dynamic barrier parameter updating scheme is discussed. The aim is to obtain a method which uses only full Newton steps, and uses large barrier parameter updates to reduce the iteration count. A worst-case iteration complexity bound of $O(\sqrt{n}L)$ of the resulting algorithm is proved.

6.1.1 The path following approach

For a given value $\mu > 0$, the point μI can be regarded as a *target point* on the central path with associated target duality gap $n\mu$. In other words if the unique centered pair $(X(\mu), S(\mu))$ can be computed, then the duality gap is equal to $n\mu$.

Path following algorithms iteratively compute $(X(\mu), S(\mu))$ approximately, followed by a decrease in the value of μ.

To derive the search directions, the notation for the primal–dual (NT) scaling as introduced in Section 3.1.1 is used. We wish to compute $(\Delta X, \Delta S)$ such that

$$(X + \Delta X)(S + \Delta S) = \mu I$$

for a given $\mu > 0$. Using the scaling matrix D defined in (3.1), this may be written as

$$(V + D_X)(V + D_S) = \mu I, \tag{6.1}$$

where $V = D^{-\frac{1}{2}} X D^{-\frac{1}{2}} = D^{\frac{1}{2}} S D^{\frac{1}{2}}$, $D_X = D^{-\frac{1}{2}} \Delta X D^{-\frac{1}{2}}$, $D_S = D^{\frac{1}{2}} \Delta S D^{\frac{1}{2}}$, and $D_V = D_X + D_S$, as before.

We now relax condition (6.1) by replacing the left hand side with its symmetric part, to obtain

$$\frac{1}{2}\left[(V+D_X)(V+D_S)+((V+D_X)(V+D_S))^T\right]=\mu I,$$

which is now linearized by neglecting the cross terms D_XD_S and D_SD_X to obtain

$$\frac{1}{2}\left((D_X+D_S)V+V(D_X+D_S)\right)=\mu I-V^2. \tag{6.2}$$

Equation (6.2) (called a Sylvester equation) has a unique symmetric solution [41], given by

$$D_V=\mu V^{-1}-V.$$

Pre and postmultiplying with $D^{\frac{1}{2}}$ yields the *Nesterov-Todd* (or NT) equations:

$$D\Delta SD+\Delta X=\mu S^{-1}-X \tag{6.3}$$

subject to

$$\left.\begin{array}{rcl}\mathbf{Tr}\,(A_i\Delta X) & = & 0,\quad i=1,\ldots,m\\ \Delta S & \in & \text{span}\,\{A_1,\ldots,A_m\}\,.\end{array}\right\} \tag{6.4}$$

6.1.2 A measure of centrality

In Section 4.6 we encountered (up to the constant $\frac{1}{2}$) the measure

$$\delta(X,S,\mu)=\frac{1}{2}\frac{1}{\sqrt{\mu}}\|D_V\|=\frac{1}{2}\left\|\sqrt{\mu}V^{-1}-\frac{1}{\sqrt{\mu}}V\right\|,$$

which was introduced by Jiang [57] (without the constant $\frac{1}{2}$). This measure generalizes the LP measure of Jansen *et al.* [51] to semidefinite programming, and will be used extensively in this chapter. It is shown in [57] that $\delta(X,S,\mu)$ is related to the directional derivative of the primal-dual barrier along the NT direction. To derive this relation, let $(\Delta X,\Delta S)$ denote the NT direction at (X,S) and let f denote the primal–dual log barrier

$$f(X,S,\mu)=\frac{1}{\mu}\mathbf{Tr}\,(XS)-\log\det(XS).$$

Further, let

$$\Delta W := \begin{bmatrix} \Delta X & 0 \\ 0 & \Delta Z \end{bmatrix}, \quad \nabla f := \begin{bmatrix} \nabla_X f & 0 \\ 0 & \nabla_Z f \end{bmatrix},$$

as in Chapter 4, such that the directional derivative is given by:

$$\begin{aligned} \langle \nabla f(X,S,\mu), \Delta W \rangle &\equiv \langle \nabla_X f(X,S,\mu), \Delta X \rangle + \langle \nabla_S f(X,S,\mu), \Delta S \rangle \\ &= \mathbf{Tr}\left(\left(\frac{1}{\mu}S - X^{-1}\right)\Delta X\right) + \mathbf{Tr}\left(\left(\frac{1}{\mu}X - S^{-1}\right)\Delta S\right) \\ &= \mathbf{Tr}\left(\left(\frac{1}{\mu}V - V^{-1}\right)D_X\right) + \mathbf{Tr}\left(\left(\frac{1}{\mu}V - V^{-1}\right)D_S\right) \\ &= \mathbf{Tr}\left(\left(\frac{1}{\mu}V - V^{-1}\right)D_V\right) \\ &= -\frac{1}{\mu}\mathbf{Tr}\left(D_V^2\right) = -4\delta^2. \end{aligned}$$

This equality shows that δ is a natural centrality measure associated with the NT direction.

6.1.3 The generic algorithm

The algorithms presented here all fit in the following framework:

Generic primal-dual logarithmic barrier algorithm

Input
A strictly feasible pair (X^0, S^0);

Parameters
Parameter $\tau < 1$ and $\mu_0 > 0$ such that $\delta(X^0, S^0, \mu_0) \leq \tau$;
An accuracy parameter $\epsilon > 0$.

begin
$X := X^0$; $S := S^0$; $\mu = \mu_0$
while $\mathbf{Tr}(XS) > \epsilon$ **do**
Compute ΔX, ΔS from (6.3) and (6.4);
$X := X + \Delta X$;
$S := S + \Delta S$;
Compute an updating parameter θ (default $\theta = \frac{1}{2\sqrt{n}}$);
$\mu := (1 - \theta)\mu$;
end
end

6.2 Feasibility of the Newton step

One can now prove the following two results which are analogous to the LP case: If $\delta < 1$ then the full Newton step is feasible, and the duality gap after the step attains its target value.

The feasibility of the Newton step is proved in the following lemma. The condition $\delta(X, S, \mu) < 1$ in the lemma is a significant improvement over the corresponding condition $\delta(X, S, \mu) < \frac{1}{2\sqrt{2}}$ derived by Jiang [57].

Lemma 6.2.1 (Condition for a feasible full Newton step) *If* $\delta := \delta(X, S, \mu) < 1$ *then the full Newton step is strictly feasible.*

Proof:

We show that the determinant of $X(\alpha)S(\alpha)$ remains positive for all $\alpha \leq 1$. One then has $X(1), S(1) \succ 0$ by Lemma 3.3.1.

To this end note that

$$\begin{aligned} X(\alpha)S(\alpha) &\sim (V + \alpha D_X)(V + \alpha D_S) \\ &= V^2 + \alpha D_X V + \alpha V D_S + \alpha^2 D_X D_S \\ &= V^2 + \alpha(\mu I - V^2) + \frac{1}{2}\alpha^2 (D_X D_S + D_S D_X) \\ &\quad + \left[\frac{1}{2}\alpha^2(D_X D_S - D_S D_X) + \frac{1}{2}\alpha(D_X V + V D_S - V D_X - D_S V)\right], \end{aligned}$$

using the Newton equation (6.2).

The matrix in square brackets in the last equation is skew–symmetric. Lemma 3.3.3 therefore implies that the determinant of $[X(\alpha)S(\alpha)]$ will be positive if the matrix

$$M(\alpha) := V^2 + \alpha(\mu I - V^2) + \frac{1}{2}\alpha^2 (D_X D_S + D_S D_X)$$

is positive definite. Since we can rewrite the expression for $M(\alpha)$ as

$$M(\alpha) = (1-\alpha)V^2 + \alpha\mu \left[I + \frac{\alpha}{\mu}\frac{1}{2}(D_X D_S + D_S D_X)\right],$$

one will have $M(\alpha) \succ 0$ if $\alpha \leq 1$ and $\left\| \frac{1}{2}(D_X D_S + D_S D_X)/\mu \right\|_2 < 1$. The last condition is easily shown to hold by using Lemma 5.3.1 and $\delta < 1$ successively:

$$\left\| \frac{1}{2}(D_X D_S + D_S D_X)/\mu \right\|_2 = \frac{1}{\mu}\left\| \frac{1}{2}(D_X D_S + D_S D_X) \right\|_2 \leq \frac{1}{4\mu}\|D_V\|^2 = \delta^2 < 1.$$

This completes the proof. □

The following result is analogous to the LP-case, and is useful in constructing μ-updating schemes.

Corollary 6.2.1 *The target duality gap is attained after one full Newton step.*

Proof:

Since

$$X(1)S(1) \sim \mu I + \frac{1}{2}\left(D_X D_S + D_S D_X\right) + \left[\frac{1}{2}(D_X D_S - D_S D_X) + \frac{1}{2}(D_X V + V D_S - V D_X - D_S V)\right] \quad (6.5)$$

one has

$$\mathbf{Tr}\ (X(1)S(1)) = \mathbf{Tr}\,(\mu I)$$

by using the orthogonality of D_X and D_S and the skew symmetry of the matrix in square brackets. □

Notation:
In what follows we denote the skew-symmetric matrix in (6.5) by M. As we will only work with full Newton steps, *i.e.* $\alpha = 1$, it will also be convenient to write $X^+ := X(1)$, $S^+ := S(1)$, etc. Finally, we can simplify the notation by defining

$$D_{XS} := \frac{1}{2}\left(D_X D_S + D_S D_X\right),$$

i.e. D_{XS} is the symmetric part of $D_X D_S$.

6.3 Quadratic convergence to the central path

We proceed to prove quadratic convergence to the target point μI. To this end we need three technical results which give information concerning the spectrum of X^+S^+. We denote the symmetrical transformation of X^+S^+ by $(V^+)^2$.

Lemma 6.3.1 *One has*

$$\lambda_{\min}\left(\left(V^+\right)^2\right) \geq \mu(1-\delta^2),$$

where $\lambda_{\min}$ denotes the smallest eigenvalue.

Proof:

From (6.5) it follows that

$$\lambda_{\min}\left(\left(V^+\right)^2\right) = \lambda_{\min}\left(\mu I + D_{XS} + M\right).$$

The skew-symmetry of M implies

$$\begin{aligned}\lambda_{\min}\left(\left(V^+\right)^2\right) &\geq \lambda_{\min}\left(\mu I + D_{XS}\right)\\ &\geq \mu - \|D_{XS}\|_2 .\end{aligned}$$

Substitution of the bound for $\|D_{XS}\|_2$ from Lemma 5.3.1 now yields:

$$\lambda_{\min}\left(\left(V^+\right)^2\right) \geq \mu - \frac{1}{4}\|D_V\|^2 = \mu\left(1-\delta^2\right),$$

which completes the proof. □

Lemma 5.3.1 gave a bound on the spectral norm of D_{XS}. We now derive a similar bound on its Frobenius norm:

Lemma 6.3.2 *One has*

$$\|D_{XS}\|^2 \leq \frac{1}{8}\|D_V\|^4 .$$

Proof:

It is trivial to verify that

$$D_X D_S + D_S D_X = \frac{1}{2}\left[(D_X + D_S)^2 - (D_X - D_S)^2\right].$$

Since D_X and D_S are orthogonal, the matrices $D_V = D_X + D_S$ and $Q_V := D_X - D_S$ have the same norm. Consequently

$$\begin{aligned}\|D_{XS}\|^2 &= \left\|\frac{1}{4}\left(D_V^2 - Q_V^2\right)\right\|^2\\ &= \frac{1}{16}\mathbf{Tr}\left(D_V^4 + Q_V^4 - 2D_V^2 Q_V^2\right)\\ &\leq \frac{1}{16}\left(\left\|D_V^2\right\|^2 + \left\|Q_V^2\right\|^2\right)\\ &\leq \frac{1}{16}\left(\|D_V\|^4 + \|Q_V\|^4\right) = \frac{1}{8}\|D_V\|^4 .\end{aligned}$$

□

The quadratic convergence result will now be proved.

Lemma 6.3.3 (Quadratic convergence) *After a feasible Newton step the distance measure satisfies*

$$\delta^+ := \delta(X^+, S^+, \mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}}.$$

Proof:

The distance measure after the full Newton step is given by

$$\begin{aligned}
\left(\delta^+\right)^2 &= \frac{1}{4\mu}\left\|\mu\left(V^+\right)^{-1} - V^+\right\|^2 \\
&= \frac{1}{4\mu}\left\|\left(V^+\right)^{-1}\left(\mu I - \left(V^+\right)^2\right)\right\|^2 \\
&\leq \frac{1}{4\mu}\lambda_{\max}^2\left(\left(V^+\right)^{-1}\right)\left\|\mu I - \left(V^+\right)^2\right\|^2.
\end{aligned}$$

We now substitute the bound from Lemma 6.3.1 to obtain

$$\left(\delta^+\right)^2 \leq \frac{1}{4\mu^2(1-\delta^2)}\left\|\mu I - \left(V^+\right)^2\right\|^2.$$

To complete the proof we show that:

$$\left\|\mu I - \left(V^+\right)^2\right\|^2 \leq \|D_{XS}\|^2. \tag{6.6}$$

In order to prove (6.6), note that

$$\begin{aligned}
\left\|\mu I - \left(V^+\right)^2\right\|^2 &= \sum_{i=1}^{n}\left[\lambda_i\left(\mu I + D_{XS} + M\right) - \lambda_i\left(\mu I\right)\right]^2 \\
&= \sum_{i=1}^{n}\left[\lambda_i\left(D_{XS} + M\right)\right]^2 \\
&= \mathbf{Tr}\left(\left(D_{XS} + M\right)^2\right).
\end{aligned}$$

Using the skew-symmetry of M one obtains

$$\begin{aligned}
\left\|\mu I - \left(V^+\right)^2\right\|^2 &= \mathbf{Tr}\left((D_{XS})^2 - MM^T\right) \\
&\leq \mathbf{Tr}\,(D_{XS})^2 = \|D_{XS}\|^2.
\end{aligned}$$

The final result now follows from Lemma 6.3.2. □

The local convergence result may be stated concisely as:

Corollary 6.3.1 *If* $\delta(X,S,\mu) < \frac{1}{\sqrt{2}}$ *then* $\delta(X^+,S^+,\mu) < \delta^2(X,S,\mu)$, i.e. *quadratic convergence to the* μ*-center is obtained. The weaker condition* $\delta(X,S,\mu) < \sqrt{\frac{2}{3}}$ *implies* $\delta(X^+,S^+,\mu) < \delta(X,S,\mu)$ *and is therefore sufficient for convergence.*

6.4 Updating the barrier parameter μ

If the current iterates are sufficiently close to the target point, say $\delta \leq \frac{1}{2}$, then the parameter μ is updated via

$$\mu^+ = (1-\theta)\mu$$

where $0 < \theta < 1$. We show that a default value of $\theta = \frac{1}{2\sqrt{n}}$ ensures that $\delta(X,S,\mu^+) \leq \frac{1}{\sqrt{2}}$. The next Newton step then again yields a feasible pair (X^+,S^+) with $\delta(X^+,S^+,\mu^+) \leq \frac{1}{2}$ due to the quadratic convergence property (Corollary 6.3.1).

We first prove a lemma which relates the distance measure after the Newton step to the measure before the step.

Lemma 6.4.1 *Let* $\delta := \delta(X,S,\mu)$ *and let* $\mathbf{Tr}\,(XS) = \mu n$. *If* $\mu^+ = (1-\theta)\mu$ *one has*

$$(\delta(X,S,\mu^+))^2 = \frac{n\theta^2}{4(1-\theta)} + (1-\theta)\delta^2.$$

Proof:

To simplify notation we introduce

$$U := \frac{1}{\sqrt{\mu}}V.$$

In terms of this notation one has

$$\begin{aligned} 4\left(\delta(X,S,\mu^+)\right)^2 &= \left\| \sqrt{1-\theta}U^{-1} - \frac{1}{\sqrt{1-\theta}}U \right\|^2 \\ &= \left\| \frac{\theta U}{\sqrt{1-\theta}} - \sqrt{1-\theta}\left(U^{-1}-U\right) \right\|^2 . \end{aligned}$$

Note that $\|U\|^2 = \mathbf{Tr}\,(U^2) = \frac{1}{\mu}\mathbf{Tr}\,(V^2) = n$. This implies that U is orthogonal to $U^{-1}-U$:

$$\mathbf{Tr}\,\left(U\left(U^{-1}-U\right)\right) = n - \|U\|^2 = 0.$$

Consequently

$$4\left(\delta(X,S,\mu^+)\right)^2 = \frac{\theta^2\|U\|^2}{1-\theta} + (1-\theta)\left\|U^{-1}-U\right\|^2 .$$

The required result now follows from the observation $\|U^{-1}-U\| = 2\delta$ together with $\|U\|^2 = n$. □

An immediate corollary of the lemma is the following: If one has a primal-dual pair (X,S) and parameter μ such that $\delta(X,S,\mu) \leq \frac{1}{2}$, and μ is updated via $\mu^+ := (1-\frac{1}{2\sqrt{n}})\mu$, then one has $\delta(X,S,\mu^+) \leq \frac{1}{\sqrt{2}}$.[1]

As discussed above, the next Newton step now yields a pair (X^+,S^+) satisfying

$$\delta(X^+,S^+,\mu^+) \leq \frac{1}{2}.$$

The algorithm therefore generates a sequence of iterates which always satisfy $\delta \leq \frac{1}{2}$. Moreover, the duality gap is reduced by a factor $(1-\frac{1}{2\sqrt{n}})$ at each iteration, since the duality gap after the Newton step equals the target duality gap.

These observations imply the following result which ascertains the polynomial convergence of the algorithm. The proof is identical to that in the LP case (see *e.g.* [98]) and is omitted here.

[1]This result can be improved to the following: If $\delta \leq \frac{1}{2}$ and μ is updated to $\mu^+ = (1 - 1/\sqrt{n+1})\mu$, then after a full Newton step with respect to μ^+ one has $\delta(X^+,S^+,\mu^+) \leq \frac{1}{2}$. We omit the proof since the improvement is only marginal — one still has $\theta = O(1/\sqrt{n})$.

Theorem 6.4.1 *If $\tau = \frac{1}{\sqrt{2}}$ and $\theta = \frac{1}{2\sqrt{n}}$ then the Primal-Dual Logarithmic barrier Algorithm with full Newton steps requires at most*

$$\left\lceil 2\sqrt{n} \log \frac{n\mu^0}{\epsilon} \right\rceil$$

iterations. The output is a primal-dual pair (X, S) satisfying $\mathbf{Tr}\,(XS) \leq \epsilon$.

6.5 Adaptive μ-updates

The short step method presented in the previous section suffers from the usual drawback that the number of iterations needed for convergence will be close to the upper bound given by Theorem 6.4.1. This is due to the small, fixed μ-updates. In an implementation it is desirable to make the largest possible updates at each iteration, albeit at the cost of extra computation.

This is the topic of this section: Given a pair (X, S) and parameter μ such that $\delta(X, S, \mu) \leq \frac{1}{2}$ we show how to obtain a larger θ such that the next Newton step with respect to $\mu^+ := (1 - \theta)\mu$ again yields a pair (X^+, S^+) satisfying $\delta(X^+, S^+, \mu^+) \leq \frac{1}{2}$. This allows much larger updates than with the strategy presented in the previous section. It is known that dynamic updating strategies can reduce the iteration count considerably in the LP case [98].

6.5.1 A condition for adaptive updates

As before, let (X^+, S^+) arise from a full Newton step from (X, S) with respect to μ. From the proofs of Lemma 6.3.1 and Lemma 6.3.3 it follows that

$$(\delta^+)^2 := \delta(X^+, S^+, \mu)^2 \leq \frac{\|D_{XS}/\mu\|^2}{4\left(1 - \|D_{XS}/\mu\|_2\right)}.$$

One will therefore have $\delta^+ \leq \tau$ for a given tolerance $\tau > 0$ if

$$\frac{\|D_{XS}/\mu\|^2}{\left(1 - \|D_{XS}/\mu\|_2\right)} \leq 4\tau^2$$

or

$$\|D_{XS}/\mu\|^2 \leq 4\tau^2\left(1 - \|D_{XS}/\mu\|_2\right). \tag{6.7}$$

Note that this condition guarantees $\|D_{XS}/\mu\|_2 \leq 1$ which in turn is sufficient to guarantee a feasible step by the proof of Lemma 6.2.1.

The updating condition (6.7) is independent of the primal-dual scaling — for any given factorization of the scaling matrix, say $D = LL^T$ (as opposed to the usual $D = D^{\frac{1}{2}}D^{\frac{1}{2}}$), one has

$$\begin{aligned} D_{XS} &:= \frac{1}{2}L^{-1}\Delta X\Delta S L + \frac{1}{2}L^T\Delta S\Delta X L^{-T} \\ &\sim \frac{1}{2}\Delta X\Delta S + \frac{1}{2}D\Delta S\Delta X D^{-1}. \end{aligned}$$

The eigenvalues of D_{XS} are therefore independent of L, and consequently so are $\|D_{XS}\|$ and $\|D_{XS}\|_2$ in (6.7).

6.5.2 An adaptive updating strategy

The matrices (D_X, D_S) depend on the parameter μ, because they correspond to the solution of

$$\frac{1}{2}\left[V(D_X + D_S) + (D_X + D_S)V\right] = \mu I - V^2. \tag{6.8}$$

In what follows D_X, D_S and $D_V = D_X + D_S$ correspond to a fixed parameter μ, and D_X^+ and D_S^+ correspond to an unknown parameter $\mu^+ = (1-\theta)\mu$, which is dependent on a variable θ. The idea is that (D_X^+, D_S^+) can be expressed in terms of (D_X, D_S) as a function of θ.

In terms of this notation the updating condition is to choose the largest value of θ such that the centrality condition (6.7) stays valid:

$$\left\|D_{XS}^+/\mu^+\right\|^2 \leq 4\tau^2\left(1 - \left\|D_{XS}^+/\mu^+\right\|_2\right). \tag{6.9}$$

In what follows we show how to approximate the largest value of θ that satisfies (6.9).

The first step is to express D_{XS}^+ in terms of D_X and D_S. To this end, note that the solution of the Newton equation (6.8) can been seen as the sum of solutions of the linear systems with right hand sides μI and V^2 respectively. The solution of

$$\frac{1}{2}\left[V(D_X + D_S) + (D_X + D_S)V\right] = \mu I,$$

will be denoted by D_X^c, D_S^c (the so-called *centering component*), and the solution of

$$\frac{1}{2}\left[V(D_X + D_S) + (D_X + D_S)V\right] = -V^2, \tag{6.10}$$

will be denoted by D_X^a, D_S^a (the so-called *affine–scaling component*). It is trivial to show that

$$\begin{aligned} D_X^+ &= (1-\theta)D_X^c + D_X^a, \\ D_S^+ &= (1-\theta)D_S^c + D_S^a. \end{aligned}$$

Using that $D_X = D_X^c + D_X^a$ and $D_S = D_S^c + D_S^a$ one has

$$\begin{aligned} D_X^+ &= (1-\theta)D_X + \theta D_X^a, \\ D_S^+ &= (1-\theta)D_S + \theta D_S^a. \end{aligned}$$

Defining $D_{XS}^a := \frac{1}{2}\left(D_X^a D_S^a + D_S^a D_X^a\right)$, we can rewrite the matrix

$$D_{XS}^+/\mu^+ := \frac{\frac{1}{2}(D_X^+ D_S^+ + D_S^+ D_X^+)}{(1-\theta)\mu}$$

as

$$\frac{D_{XS}^+}{\mu^+} = \frac{1-\theta}{\mu}D_{XS} + \frac{1}{2\mu}\theta\left(D_X D_S^a + D_S D_X^a + D_X^a D_S + D_S^a D_X\right) + \frac{\theta^2}{\mu(1-\theta)}D_{XS}^a. \tag{6.11}$$

Note that the entries of $(1-\theta)D_{XS}^+/\mu^+$ are quadratic functions of θ. The updating condition (6.9) can be multiplied by $(1-\theta)$ and rewritten as

$$4\tau^2(1-\theta)^2 - \left\|(1-\theta)D_{XS}^+/\mu^+\right\|^2 \geq 4\tau^2(1-\theta)\left\|(1-\theta)D_{XS}^+/\mu^+\right\|_2. \tag{6.12}$$

Note that the left hand side is a fourth order polynomial in θ, while the right hand side can be a more general nonlinear function of θ. Condition (6.12) is guaranteed to hold if we replace the right hand side expression by a larger value. To this end, we use the triangle inequality

$$\begin{aligned} \left\|(1-\theta)D_{XS}^+/\mu^+\right\|_2 &\leq \frac{1-\theta}{\mu}\left\|D_{XS}\right\|_2 \\ &+ \frac{1}{2\mu}\theta(1-\theta)\left\|D_X D_S^a + D_S D_X^a + D_X^a D_S + D_S^a D_X\right\|_2 \\ &+ \frac{\theta^2}{\mu}\left\|D_{XS}^a\right\|_2. \end{aligned}$$

If we replace the right hand side of (6.12) by this expression, the updating condition reduces to finding the largest root of a fourth order polynomial — an analytically solvable problem.

Remark:
If the current iterates are perfectly centered with respect to μ, then $D_X = D_S = 0$, $D_X^a = -D_X^c$, and $D_S^a = -D_S^c$. It follows that equation (6.11) simplifies to

$$D_{XS}^+/\mu^+ = \frac{\theta^2}{\mu(1-\theta)} D_{XS}^a.$$

In this case the original updating condition (6.9) can be solved, since $(1-\theta)\left\|D_{XS}^+/\mu^+\right\|_2$ is now a quadratic function of θ. This observation is of interest as the starting point can be assumed perfectly centered by embedding the original primal-dual problem pair into a larger problem, where a feasible starting point on the central path is known, as was shown in Chapter 2. The first μ-update can therefore be done by solving (6.9). In subsequent iterations, centering steps will be needed to approximate this feature. The effect of centering steps on the updating strategy will be illustrated in the next section.

6.5.3 Numerical examples

We investigate a small example $(n = 3, m = 3)$ where different updating strategies are used. The example problem is given by

$$A_1 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 6 & 4 \\ 3 & 4 & -2 \end{bmatrix},$$

with feasible starting solution

$$X^0 = \begin{bmatrix} 1 & 0.1 & 0.1 \\ 0.1 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix}, \; S^0 = \begin{bmatrix} 1.05 & 0.005 & 0 \\ 0.005 & 0.95 & 0 \\ 0 & 0 & 1.1 \end{bmatrix}.$$

First the fixed default value of $\theta = 1/(2\sqrt{n})$ is used. The plot in Figure 6.1 shows the smallest and largest eigenvalues of the iterates XS, and the solid line indicates

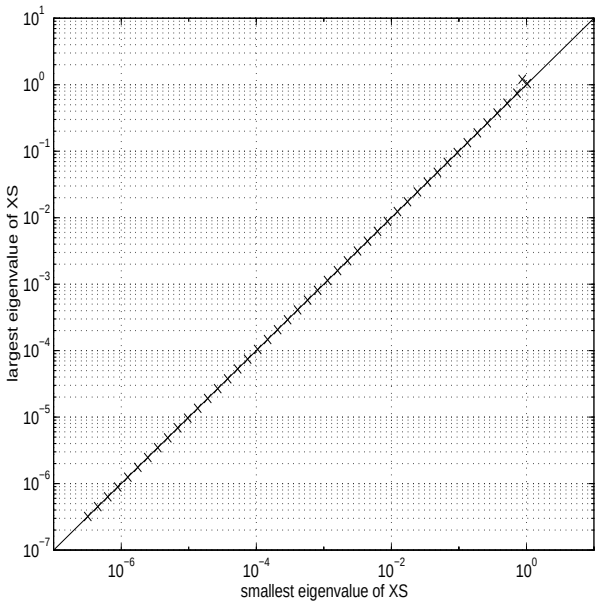

Figure 6.1: Iterates for a fixed updating scheme. The solid line corresponds to the central path.

the central path (where the smallest and largest eigenvalues are equal). In this example 45 iterations are required to meet the convergence condition $\mathbf{Tr}\,(XS) \leq 10^{-6}$. This is close to the upper bound of 52 iterations given by Theorem 6.4.1 for the example.

If the adaptive updating strategy is employed, the iterates are as shown in Figure 6.2. Only eight iterations are needed for convergence now, and the values of θ at each iteration are as shown in Table 6.1. Note that the proximity after step is typically well below $\frac{1}{2}$, which shows that the update made using the updating condition is not the largest possible.

Finally, we investigate the effect of an extra centering step at each iteration. In other words, a step $(\Delta X, \Delta S)$ satisfying:

$$D\Delta S D + \Delta X = \mu S^{-1}$$

subject to

$$\begin{aligned} \mathbf{Tr}\,(A_i \Delta X) &= 0, \quad i = 1, \ldots, m \\ \Delta S &\in \text{span}\,\{A_1, \ldots, A_m\} \end{aligned}$$

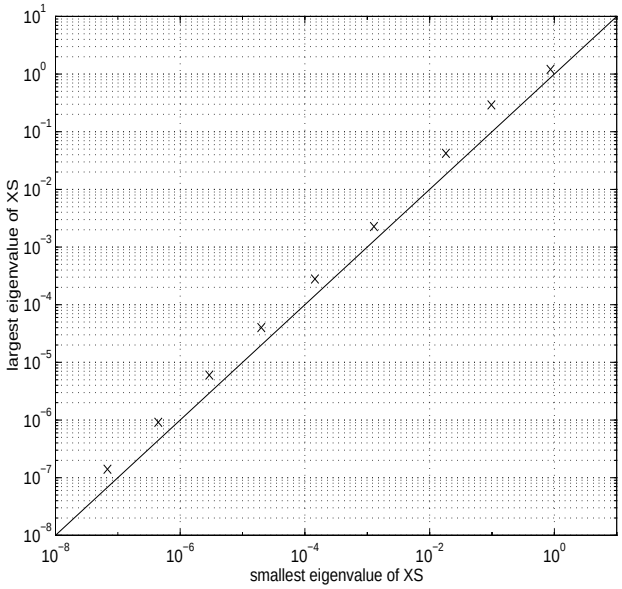

Figure 6.2: Iterates for the adaptive updating scheme. The solid line corresponds to the central path.

is made before a new μ update is done using the adaptive updating strategy. The plot in Figure 6.3 shows that the result is fast asymptotic convergence. Unfortunately there is no theoretical justification for superlinear convergence properties at this time.

Only four μ-updates are needed for convergence. The values of θ in subsequent updates, as well as values of the proximity δ before and after centering are shown in Table 6.2. It is a well-known observation for LP that centering steps within the region of quadratic convergence are extremely efficient (see *e.g.* [98]). This is also true for the example in Table 6.2.

6.6 The analysis of related methods

In this section we show how to perform the worst-case analysis of two other well-known primal–dual methods, using the tools developed here and in Chapter 4.

The first method is the potential reduction method of Nesterov and Todd [82],

iteration	θ	δ	gap
0	–	0.1163	0.3101
1	0.8045	0.4024	6.061×10^{-1}
2	0.8606	0.3115	8.447×10^{-2}
3	0.9380	0.2066	5.229×10^{-3}
4	0.8804	0.2372	6.252×10^{-4}
5	0.8594	0.2536	8.790×10^{-5}
6	0.8512	0.2582	1.307×10^{-5}
7	0.8478	0.2598	1.989×10^{-6}

Table 6.1: Sequence of updating parameters θ for the adaptive updating scheme. The centrality measure δ and duality gap after each step is also shown.

which already appeared in Chapter 4. The second is the long step primal-dual method by Jiang [57].

6.6.1 The method of Nesterov and Todd

This potential reduction method was introduced in Section 4.8. It can also be interpreted as a path-ollowing method using the dynamic μ-updates. Recall that the search direction is computed from

$$D\Delta SD + \Delta X = \mu S^{-1} - X, \tag{6.13}$$

with

$$\mu = \frac{\mathbf{Tr}\,(XS)}{(n + \nu\sqrt{n})} \tag{6.14}$$

for some $\nu \geq 1$. This choice of μ can be seen as a dynamic updating strategy — the target duality gap $n\mu$ is determined by the current gap $\mathbf{Tr}\,(XS)$, and the size of the update depends on the value of ν.

A single step is taken with respect to the target point, after which it is updated. The step length is determined by a plane search of the Tanabe-Todd-Ye potential

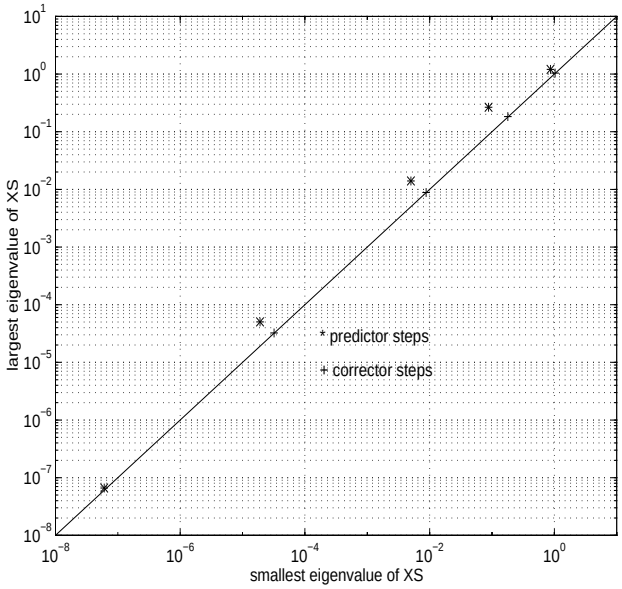

Figure 6.3: Iterates using adaptive updates followed by centering steps. Both centering and updating (predictor) steps are shown. The solid line corresponds to the central path.

function

$$\phi(X, S) := (n + \nu\sqrt{n}) \log \mathbf{Tr}\,(XS) - \log \det(XS) - n \log n,$$

as described in Section 4.4. (The plane search is necessary, as a full Newton step will not be feasible in general, because of the large updates.)

Recall from Chapter 4 that if ϕ can be reduced by an absolute constant c_{red} at each iteration, then at most

$$\left\lceil \frac{\sqrt{n}\nu L + \Psi(X^0, S^0)}{c_{red}} \right\rceil \tag{6.15}$$

iterations are required to compute a strictly feasible pair (X^*, S^*) satisfying $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$, where $L = \log \frac{\mathbf{Tr}\,(X^0S^0)}{\epsilon}$, as before.

The algorithm can be stated as follows.

Iteration	δ	θ	gap
0	0.1163	–	3.101
1c	0.0050	–	3.101
1p	0.4157	0.8235	0.5474
2c	0.0075	–	0.5474
2p	0.3381	0.9468	2.910×10^{-2}
3c	0.0012	–	2.910×10^{-2}
3p	0.3378	0.9957	1.265×10^{-4}
4c	0.0012	–	1.265×10^{-4}
4p	0.0269	0.9987	1.647×10^{-7}

Table 6.2: Iterations of the predictor-corrector type method. Centering steps are indicated by 'c' and μ-update steps by 'p'. The values of the proximity δ and duality gap after each step are shown, as well as the update parameter θ at each μ-update.

Nesterov–Todd potential reduction method

Input
A strictly feasible starting pair (X^0, S^0);

Parameters
An accuracy parameter $\epsilon > 0$;
A potential parameter $\nu \geq 1$.
begin
$X := X^0$; $S := S^0$;
while $\mathbf{Tr}\,(XS) > \epsilon$ **do**
$\mu = \mathbf{Tr}\,(XS)/(n + \nu\sqrt{n})$;
Compute $\Delta X, \Delta S$ from (6.13) and (6.4) ;
Find $(\alpha, \beta) = \text{argmin}\ \phi(X + \alpha\Delta X, S + \beta\Delta S)$;
$X := X + \alpha\Delta X$; $S := S + \beta\Delta S$;
end
end

We now prove the complexity bound, by showing that ϕ can always be reduced by an absolute constant.

Lemma 6.6.1 *The potential ϕ can be decreased along the search direction of the Nesterov-Todd potential reduction algorithm by at least:*

$$\Delta\phi \geq \psi(\xi(\delta))$$

where the function ψ is defined in (4.5), and

$$\xi(\delta) := \frac{4\delta}{2\delta + \sqrt{4 + 4\delta^2}}.$$

Proof:

By Lemma 4.3.2 one has

$$\Delta\phi \geq -\alpha(n + \nu\sqrt{n})\mathbf{Tr}\left(\frac{VD_V}{\|V\|^2}\right) + \alpha\mathbf{Tr}\left(V^{-1}D_V\right) - \psi(-\alpha h) \qquad (6.16)$$

where h is defined by

$$h := \sqrt{\left\|X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}\right\|^2 + \left\|Z^{-\frac{1}{2}}\Delta Z Z^{-\frac{1}{2}}\right\|^2}.$$

Substituting the Newton direction D_V from (4.9) into (6.16) yields

$$\begin{aligned}
\Delta\phi &\geq -\alpha(n + \nu\sqrt{n})\mathbf{Tr}\left(\frac{V(\mu V^{-1} - V)}{\|V\|^2}\right) + \alpha\mathbf{Tr}\left(V^{-1}(\mu V^{-1} - V)\right) - \psi(-\alpha h) \\
&= \alpha(n + \nu\sqrt{n})\left[1 - \frac{n\mu}{\|V\|^2}\right] + \alpha\mathbf{Tr}\left(\sqrt{\mu}V^{-1} - \frac{1}{\sqrt{\mu}}V\right)^2 - \alpha\mathbf{Tr}\left(\frac{1}{\mu}V^2 - I\right) - \psi(-\alpha h) \\
&= 2\alpha n - \alpha n\left(\frac{\|V\|^2}{n\mu} + \frac{n\mu}{\|V\|^2}\right) + \alpha\nu\sqrt{n}\left(1 - \frac{n\mu}{\|V\|^2}\right) + 4\alpha\delta^2 - \psi(-\alpha h),
\end{aligned}$$

where we have used the definition of δ. Recall that the updating strategy (6.14) is such that

$$\mu = \frac{\|V\|^2}{n + \nu\sqrt{n}}. \qquad (6.17)$$

Substituting eq. (6.17) into the bound for $\Delta\phi$ yields

$$\begin{aligned}\Delta\phi &\geq 2\alpha n - \alpha n\left(1+\frac{\nu}{\sqrt{n}}+\frac{n}{n+\nu\sqrt{n}}\right)+\alpha\nu\sqrt{n}\left(1-\frac{n}{n+\nu\sqrt{n}}\right)+4\alpha\delta^2-\psi(-\alpha h)\\ &= 2\alpha n-\alpha n-\alpha n\left(\frac{n+\nu\sqrt{n}}{n+\nu\sqrt{n}}\right)+4\alpha\delta^2-\psi(-\alpha h)\\ &= 2\alpha n-2\alpha n+4\alpha\delta^2-\psi(-\alpha h)=4\alpha\delta^2-\psi(-\alpha h).\end{aligned}$$

The last expression is maximized by

$$\alpha = 1/h - 1/(4\delta^2+h), \tag{6.18}$$

which implies

$$\Delta\phi \geq f(\alpha^*) = (4\delta^2/h) - \log(1+4\delta^2/h) = \psi(4\delta^2/h).$$

We saw in Section 4.6 that this inequality will still hold if $4\delta^2/h$ is replaced by the larger value

$$\xi(\delta) := \frac{2\delta}{\delta+\sqrt{1+\delta^2}}.$$

One therefore has that $\Delta\phi \geq \psi(\xi(\delta))$, which completes the proof. □

Loosely speaking, we can always reduce ϕ by an absolute constant if δ is large enough. Moreover, we know from Lemma 6.4.1 that after an update $\mu = (1-\theta)\mathbf{Tr}\,(XS)$, δ will be bounded from below by

$$\left(\delta(X,S,\mu^+)\right)^2 \geq \frac{n\theta^2}{4(1-\theta)}.$$

It is easy to check that the μ-updating strategy in (6.14) corresponds to

$$\theta = \frac{\nu}{\sqrt{n}+\nu},$$

and therefore by straightforward calculation one has

$$\left(\delta(X,S,\mu^+)\right)^2 \geq \frac{\nu^2 n+\nu^3\sqrt{n}}{4\left(n+\nu^2+2\nu\sqrt{n}\right)}.$$

Using $n \geq 2$ and $\nu \geq 1$ we therefore have $\delta(X, S, \mu^+) \geq 0.38$. Substituting this value into the bound from Lemma 6.6.1 yields $\Delta\phi > 0.1$. This bound for $\Delta\phi$ is rather pessimistic — it is easy to check that for $\nu \geq 10$, one has $\Delta\phi \geq 0.27$, for example.

The algorithm therefore has the following complexity, by (6.15).

Theorem 6.6.1 *The Nesterov-Todd potential reduction algorithm requires at most*

$$\left\lceil \frac{\sqrt{n}\nu L + \Psi(X^0, S^0)}{0.1} \right\rceil$$

iterations to compute a strictly feasible pair (X^*, S^*) *satisfying* $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$. □

This is essentially the same bound as derived by Nesterov and Todd [82].

6.6.2 A long step path-following method

This algorithm performs damped Newton steps with respect to a given μ until the condition $\delta(X, S, \mu) \leq \frac{1}{\sqrt{2}}$ is met. These steps are termed *inner iterations*. Only then is the parameter μ updated via $\mu^+ = (1 - \theta)\mu$ (*outer iteration*). The step lengths are determined by line searches of the primal-dual logarithmic barrier function

$$f(X, S, \mu) = \frac{1}{\mu}\mathbf{Tr}\,(XS) - \log\det(XS).$$

More formally, the algorithm is as follows.

Long step primal-dual path following method

Input
A strictly feasible starting pair (X^0, S^0);

Parameters
A centering parameter $\tau_0 > 0$ (default $\tau_0 = \frac{1}{\sqrt{2}}$);
An accuracy parameter $\epsilon > 0$;
An updating parameter $\theta < 1$;
begin
$X := X^0$; $S := S^0$;
while $\mathbf{Tr}\,(XS) > \epsilon$ **do**
if $\delta(X, S, \mu) \leq \tau_0$ **do** (*outer iteration*)
$\mu := (1 - \theta)\mu$;
else if $\delta(X, S, \mu) > \tau_0$ **do** (*inner iteration*)
Compute $\Delta X, \Delta S$ from (6.3) and (6.4) ;
Find α such that $f(X, S, \mu) - f(X + \alpha\Delta X, S + \alpha\Delta S, \mu) \geq 1/6$;
(A suitable default choice for α is given by (6.18).)
$X := X + \alpha\Delta X$, $S := S + \alpha\Delta S$;
end
end
end

We proceed to give a simple analysis of the long step method to obtain a complexity bound that is better in terms of θ than the best known (even for the special LP case). We derive the bound for the default step lengths from (6.18), *i.e.* we show that the default step length reduces ϕ by an absolute constant. To avoid confusion, note that ϕ is only used in the analysis and *not* by the algorithm.

Lemma 6.6.2 *The potential ϕ can be decreased along the search direction of the long step algorithm of Jiang by at least:*

$$\Delta\phi \geq \psi(\xi(\delta))$$

where the function ψ is defined in (4.5), and

$$\xi(\delta) := \frac{4\delta}{2\delta + \sqrt{4 + 4\delta^2}},$$

provided that $\nu = \frac{\theta}{1-\theta}\sqrt{n}$.

Proof:

By Lemma 6.6.1 one has

$$\Delta\phi \geq 2\alpha n - \alpha n \left(\frac{\|V\|^2}{n\mu} + \frac{n\mu}{\|V\|^2} \right) + \alpha\nu\sqrt{n} \left(1 - \frac{n\mu}{\|V\|^2} \right) + 4\alpha\delta^2 - \psi(-\alpha h). \tag{6.19}$$

At the start of the outer iteration, μ is updated via $\mu \leftarrow (1-\theta)\mu$, for some fixed $\theta \in (0,1)$. The value μ now remains fixed during the inner iteration phase, *i.e.* until the last iterates (X, S) satisfy $\delta(X, S, \mu) \leq \frac{1}{\sqrt{2}}$. During each inner iteration one will have $n\mu = (1-\bar{\theta})\mathbf{Tr}\,(XS)$ for some $\bar{\theta} \leq \theta$. Let us now investigate the potential reduction during the inner iteration phase. Letting $n\mu = (1-\bar{\theta})\|V^2\|$, (6.19) becomes

$$\Delta\phi \geq 2\alpha n - \alpha n \left(\frac{1}{1-\bar{\theta}} + 1 - \bar{\theta} \right) + \alpha\nu\sqrt{n}\bar{\theta} + 4\alpha\delta^2 - \psi(-\alpha h).$$

If we can guarantee

$$\nu \geq \frac{\sqrt{n}\bar{\theta}}{1-\bar{\theta}}, \tag{6.20}$$

then the bound becomes

$$\Delta\phi \geq 4\alpha\delta^2 - \psi(-\alpha h).$$

The value for ν in (6.20) will be maximal when $\bar{\theta} = \theta$, *i.e.* at the start of the inner iteration phase. We can therefore always use the value

$$\nu = \frac{\sqrt{n}\theta}{1-\theta}$$

and still guarantee the potential reduction. As in Lemma 6.6.1, we therefore have that a step length α from (6.18) reduces ϕ by at least $\xi(\delta)$. □

The lemma implies that $\Delta\phi \geq 1/6$ while $\delta > 1/\sqrt{2}$, *i.e.* during the inner iteration phase. This yields the following complexity bound, by (6.15).

Theorem 6.6.2 *The long step primal–dual path following algorithm requires at most*

$$\left\lceil 6\left(n\left(\frac{\theta}{1-\theta}\right)L + \Psi(X^0, S^0)\right)\right\rceil$$

iterations to compute a strictly feasible pair (X^*, S^*) *satisfying* $\mathbf{Tr}\,(X^*S^*) \leq \epsilon$, *if the default step length from (6.18) is used.*

□

This bound is better in terms of θ than the best known bound (even in the LP case), where the bound is proportional to $\frac{\theta}{(1-\theta)^2}$ (see *e.g.* [98], §7.8.3).

6.7 Bibliographical notes

Two different issues which enjoy some attention in the literature are considered in this section.

The difficulties surrounding superlinear convergence analysis for algorithms using the NT direction are reviewed first. The discussion revolves around a counterexample devised by Kojima *et al.* [63].

The second issue concerns the possibility of *weighted path following methods*, *i.e.* algorithms which follow generalizations of the central path.

6.7.1 Superlinear convergence

The algorithm with centering steps in Section 6.5.3 can be viewed as a predictor-corrector method, though not in the traditional sense — in the Mizuno-Todd-Ye [72] type predictor corrector methods the predictor direction is fixed as the primal-dual affine scaling direction. The predictor step length is then chosen to be the maximal step which yields a pair (X, S) satisfying

$$\left\| I - \frac{n}{\mathbf{Tr}\,(V^2)}V^2 \right\| \leq \frac{1}{2}$$

(see *e.g.* [106]). This step length can be computed analytically as the solution of a fourth order polynomial. The predictor step is then followed by a single centering step.

The following example by Kojima *et al.* [63] shows that superlinear convergence cannot be achieved using the NT direction in a Mizuno-Todd-Ye predictor-corrector scheme — it is necessary to use more than one centering step.

A necessary condition for superlinear convergence of predictor-corrector methods is as follows: the maximal feasible step length along the predictor direction must approach one. The example below does not satisfy this condition.

Consider the problem:

$$\min \mathbf{Tr}\,(CX)$$

subject to

$$\begin{aligned} \mathbf{Tr}\,(A_i X) &= b_i, \quad i = 1, 2, \\ X &\succeq 0, \end{aligned}$$

where

$$A_1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 0 \end{pmatrix},$$

which has the associated dual problem (D):

$$\max b^T y$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} y_i A_i + S &= C \\ S &\succeq 0. \end{aligned}$$

The problem pair (P),(D) has a strictly complementary solution pair

$$X^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad y^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consider the sequence of feasible iterates $(X_k, S_k, y_k) \to (X^*, S^*, y^*)$ defined by

$$X_k := \begin{pmatrix} 1 & \epsilon_k \\ \epsilon_k & \epsilon_k \end{pmatrix}, \quad S_k := \begin{pmatrix} (1+c)\epsilon_k & -\sqrt{c\epsilon_k} \\ -\sqrt{c\epsilon_k} & 1 + 2\sqrt{c\epsilon_k} \end{pmatrix}, \quad y_k := \begin{pmatrix} (1+c)\epsilon_k/2 \\ \sqrt{c\epsilon_k} \end{pmatrix},$$

where $\epsilon_k \to 0$ and $c > 0$. Here the values $c := \frac{1}{32}$ and $\epsilon_k := 10^{-k/10}$ ($k = 30, \ldots, 80$) will be used. We denote the symmetrical transformation of $X_k S_k$ by V_k.

We investigate the centrality of the iterates in terms of two different centrality measures. The first is the standard measure $\frac{1}{2}\|\frac{V^2}{\mu} - I\|$ where $\mu = \mathbf{Tr}\,(V^2)/n$. The second measure is the one used in this chapter, $\frac{1}{2}\|\mu^{-\frac{1}{2}}V - \mu^{\frac{1}{2}}V^{-1}\|$. It seems clear from Figure 6.4 that both proximity measures remain bounded from above for the sequence of iterates. In particular, all the iterates lie in the region of quadratic convergence of the central path. The maximal feasible step length to the boundary along the NT predictor direction does

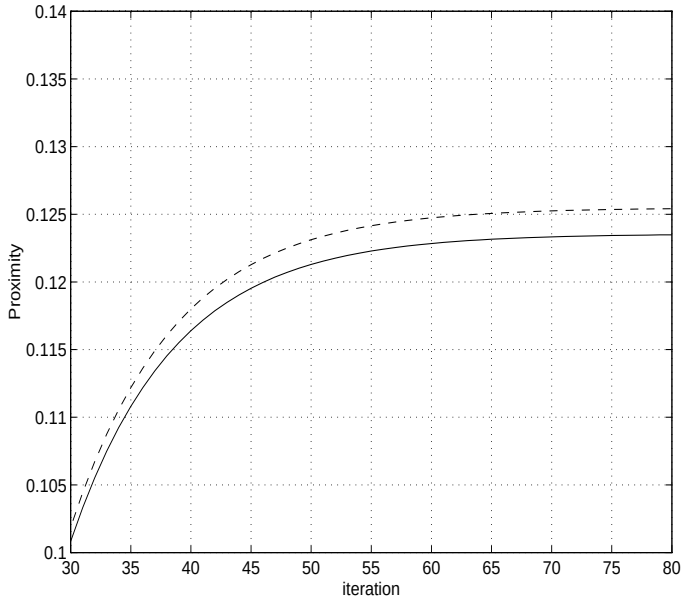

Figure 6.4: The two centrality measures $\frac{1}{2}\|\frac{V_k^2}{\mu} - I\|$ (solid line) and $\frac{1}{2}\|\mu^{-\frac{1}{2}}V_k - \mu^{\frac{1}{2}}V_k^{-1}\|$ (dashed line) indicate that the given sequence of iterates (X^k, S^k) lie in the region of quadratic convergence of the central path.

not approach one along the sequence of iterates, as is seen from Figure 6.5. In order to guarantee superlinear convergence, it is therefore *not* sufficient to take predictor steps if, say, $\delta(XS) \leq \frac{1}{\sqrt{2}}$. It is necessary to require 'increasing centrality' as $\mu \to 0$.

Superlinear convergence has been proved for the Mizuno-Todd-Ye predictor-corrector scheme under assumptions of a strictly complementary solution and increasingly centered iterates [63, 70]. The requirement for 'increasing centrality' necessitates extra centering steps in the formulation of a superlinearly convergent algorithm, while extra centering

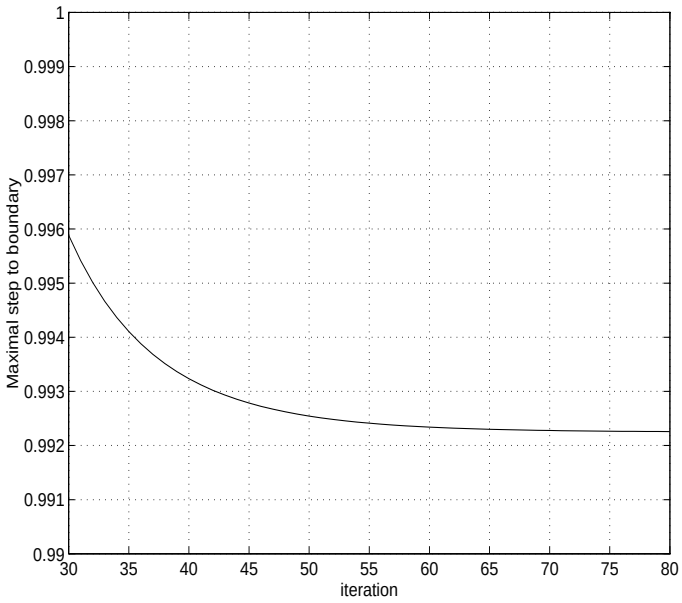

Figure 6.5: The step length to the boundary along the N-T predictor direction does not converge to one along the sequence of iterates (X_k, S_k).

steps are never done in practice.

6.7.2 Concepts of weighted centers

In this chapter central path following methods were described. For LP, this notion has been extended to the so-called *weighted central path*, see *e.g.* [46].

In the SDP case there are two such notions, which will briefly be discussed here. The first is due to Monteiro and Pang [75].

Theorem 6.7.1 (Weighted centers I) *Let (P) and (D) be strictly feasible. For each* $W \succ 0$ *there exists a unique strictly feasible pair* (X, S)*, such that*

$$\frac{1}{2}(XS + SX) = W.$$

□

De Klerk *et al.* [25] have noted that this unique pair minimizes the *weighted potential function*:

$$f_W(X,S) := \mathbf{Tr}\,\left(XSW^{-1}\right) - \log\det\left(\frac{1}{2}(XS+SX)W^{-1}\right)$$

over the set

$$\{(X,S) \in \mathcal{P}\times\mathcal{D} \;:\; XS+SX \succeq 0\}\,.$$

It is also shown in [25] that this potential function can be used to construct so-called *target following* algorithms in the LP case. This analysis is given in Appendix B. It is not yet clear how to extend this analysis to semidefinite programming.

The second 'weighted centers' concept is due to Sturm and Zhang [105], and is as follows.

Theorem 6.7.2 (Weighted centers II) *To each set set of positive 'target eigenvalues', say $\omega_1 \geq \ldots \geq \omega_n > 0$, corresponds a strictly feasible pair (X,S), in the sense that*

$$\lambda_i(XS) = \omega_i, \quad i = 1,\ldots,n.$$

This pair (X,S) is not unique in general. □

A comparison of the two concepts may also be found in [105].

Chapter 7

Conclusions

A summary of the main results contained in this thesis is given here. Some suggestions for future research are also included.

7.1 Overview of results and future research directions

Chapter 2 contained a systematic approach for the investigation of complexity issues for SDP. In the first instance, results on the convergence of the central path were proved. These included the results of De Klerk *et al.* [22, 24] (*i.e.* limit points of the central path are maximally complementary), as well as simplified proofs of subsequent results by Goldfarb and Scheinberg [37] (*i.e.* the central path converges to the analytic center of the optimal set).

In the second instance, the concept of self-dual embeddings was investigated; this yielded a general algorithmic framework for solving SDP problems with no regularity assumptions. The main results were:

- the existence of a complementary optimal pair can be established (or excluded), and strong infeasibility can be detected (or excluded) in polynomial time, if

  [i] the embedding problem is known to have a strictly complementary solution;

[ii] a certain condition number of the embedding problem is known;

- if no strictly complementary solution exists, then additional information is needed on the convergence rate of the central path;
- by using Extended Lagrange-Slater duals in the embedding, weak infeasibility and finite optimal values can be detected asymptotically, subject to the above conditions.

The treatment in Chapter 2 solves an open problem posed by Ramana and Pardalos [95], namely how to incorporate ELSD duals in an infeasible start algorithmic framework. Moreover, it casts some light on the most important question in complexity of SDP, namely: can feasibility of a given SDP problem be decided in polynomial time? (see [95] for a discussion of this problem.) In view of this discussion, the two open problems posed in Chapter 2 must be regarded as important avenues for future research.

In Chapter 3, two primal–dual affine–scaling methods are extended from LP to SDP, based on the results of De Klerk *et al.* in [27]. Affine–scaling methods have been of great theoretical interest since it became apparent that Karmarkar's [58] famous algorithm was closely related to the primal affine–scaling method of Dikin [29] from 1967. The extensions in Chapter 3 are also significant in the light of failure of some related methods: The primal affine–scaling method fails for SDP [78], as do some primal–dual affine–scaling methods for SDP where other primal–dual scalings are used [79]. To summarize the results:

- both the classical primal–dual affine–scaling algorithm of Monteiro et al [74], as well as the Dikin-type primal–dual affine–scaling algorithm of Jansen *et al.* [49] were successfully extended from LP to SDP;
- the use of the Nesterov-Todd scaling is crucial in the extensions from LP to SDP; the related algorithms using the so-called HRWV/KSH/M or AHO scalings (see Section 3.6) fail.

In Chapter 4, the analysis of the primal–dual affine–scaling methods was used to formulate implementable variants of these methods, based on De Klerk *et al.* [28]. This resulted in potential reduction algorithms which function as long step predictor–corrector methods. A numerical comparison was presented with the potential reduction method of Nesterov and Todd [85], as implemented by Vandenberghe and Boyd [110]. The new algorithms perform competitively on three well-known classes of SDP problems; on education testing problems the new methods

are slightly superiour, but their performances worsen on logarithmic Chebychev approximation and matrix norm minimization problems. In summary:

- two new potential reduction methods were formulated and analysed. In this way the analysis of potential reduction methods was linked to that of primal–dual affine–scaling methods;
- the new methods were shown to perform competitively on three sets of test problems.

The primal path–following (log-barrier method) was analysed in Chapter 5, based on the paper [43]. The two main contributions were

- simplified proofs were given in obtaining the complexity results;
- it was shown how to use primal methods to solve the embedding problems of Chapter 2 if the original problem is in *symmetric form*. In this way purely primal methods function as infeasible-start primal–dual algorithms, which may restore some computational interest in these algorithms.

Primal–dual path following methods were studied in Chapter 6, based on results by De Klerk *et al.* [26]. In particular, results pertaining to a centrality measure introduced by Jiang [57] were refined and extended, namely:

- a weaker condition for a full Newton step in terms of the new centrality measure was established;
- quadratic convergence of iterates to target points on the central path was proved;

Moreover, adaptive barrier parameter updating strategies were formulated and illustrated. This resulted in algorithms which only use full Newton steps but do not suffer from the high iteration count normally associated with short step algorithms. It seems worthwhile to experiment with serious implementations of these algorithms in future, and to study (possible) superlinear convergence properties. To prove superlinear convergence is still an open problem — even in the special case of LP (see [98]).

Furthermore, a new analysis was given of two related methods, namely the potential reduction method by Nesterov and Todd [85], and the long step method by Jiang [57]. This new analysis utilizes the analytic tools developed in this thesis.

The last contribution in Chapter 6 was a discussion of concepts of weighted centers for SDP. A nonconvex weighted potential function was suggested which could possibly be used in constructing weighted path following algorithms for SDP. To perform this analysis is still an open problem, but it can be done for the special case of LP, as is shown in Appendix B. This analysis is based on results of De Klerk *et al.* [25].

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In conclusion, SDP is one of the fastest growing branches of mathematical programming. The development of interior point algorithms for this class of problems has opened the way to solve well-known problems efficiently. (A partial list of such problems was given in Chapter 1.) It has also stimulated researchers to explore new applications, and many exciting new results on the quality of SDP relaxations of combinatorial and global optimization problems have been forthcoming. It would seem that the lid has barely been scratched on both the complexity theory and applications of semidefinite programming. The study of interior point algorithms for SDP is also not exhausted yet — new search directions are still being proposed at a furious rate, and the list of SDP codes is growing. Although this thesis was biased towards algorithms using the Nesterov-Todd direction, only time will tell if it will really become the method of choice.

Appendix A

Matrix calculus and linear algebra lemmas

A.1 Calculus results

The first two lemmas may be used to derive the gradient of a barrier function like the primal barrier,

$$f_p(X) = \frac{1}{\mu}\mathbf{Tr}\,(CX) - \log\det(X),$$

or the primal–dual variant

$$f_{pd}(X,S) := \frac{1}{\mu}\mathbf{Tr}\,(XS) - \log\det(XS).$$

Lemma A.1.1 *Let* $f : \mathrm{int}\,(\mathcal{S}_n^+) \mapsto \mathbb{R}$ *be given by*

$$f(X) = \log\det X,$$

Denoting

$$\nabla f(X) := \begin{bmatrix} \frac{\partial f(X)}{\partial x_{11}} & \cdots & \frac{\partial f(X)}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(X)}{\partial x_{n1}} & \cdots & \frac{\partial f(X)}{\partial x_{nn}} \end{bmatrix},$$

one has $\nabla f(X) = X^{-1}$.

Proof:

Let $X \in \text{int}\,(\mathcal{S}_n^+)$ be given and let $H \in \mathcal{S}_n$ be such that $X + H \in \text{int}(\mathcal{S}_n^+)$. One has

$$\begin{aligned} f(X+H) - f(X) &= \log\det(X+H) - \log\det(X) \\ &= \log\det\left(X^{-1}(X+H)\right) \\ &= \log\det\left(I + X^{-\frac{1}{2}} H X^{-\frac{1}{2}}\right). \end{aligned}$$

By the arithmetic-geometric inequality applied to the eigenvalues of $X^{-\frac{1}{2}} H X^{-\frac{1}{2}}$ one has

$$\begin{aligned} \log\det\left(I + X^{-\frac{1}{2}} H X^{-\frac{1}{2}}\right) &\leq \log\left(\frac{1}{n}\mathbf{Tr}\,\left(I + X^{-\frac{1}{2}} H X^{-\frac{1}{2}}\right)\right)^n \\ &= n\log\left(\frac{1}{n}\mathbf{Tr}\,\left(I + X^{-\frac{1}{2}} H X^{-\frac{1}{2}}\right)\right) \\ &= n\log\left(1 + \frac{1}{n}\mathbf{Tr}\,\left(X^{-\frac{1}{2}} H X^{-\frac{1}{2}}\right)\right). \end{aligned}$$

Using the well-known inequality $\log(1+t) \leq t$ we arrive at

$$f(X+H) - f(X) \leq \mathbf{Tr}\,\left(X^{-\frac{1}{2}} H X^{-\frac{1}{2}}\right) = \langle X^{-1}, H\rangle.$$

This shows that X^{-1} is a subgradient of f at X. Since f is assumed differentiable, the subgradient is unique and equals the gradient $\nabla f(X)$. □

The proof of the next result is trivial.

Lemma A.1.2 *Let* $f : \text{int}\,(\mathcal{S}_n^+) \mapsto \mathbb{R}$ *be given by*

$$f(X) = \mathbf{Tr}\,(CX),$$

where $C \in \mathcal{S}_n$. *One has* $\nabla f(X) = C$.

The following result is used to derive the Hessian of the log-barrier function $f_{bar}(X) = -\log\det(X)$.

Lemma A.1.3 *Let* $f : \text{int}\,(\mathcal{S}_n^+) \mapsto \mathbb{R}$ *be given by*

$$f(X) = \log\det X.$$

If $\nabla^2 f$ *denotes the derivative of* $\nabla f : X \mapsto X^{-1}$ *with respect to* X*, then* $\nabla^2 f(X)$ *is the linear operator which satisfies*

$$\nabla^2 f(X)H = -X^{-1}HX^{-1}, \quad \forall H \in \mathcal{S}_n,$$

for a given invertible X*.*

Proof:

Let $\mathrm{L}(\mathcal{S}_n, \mathcal{S}_n)$ denote the space of linear operators which map $\mathcal{S}_n$ to $\mathcal{S}_n$. The Frechet derivative of ∇f is defined as the (unique) function $\nabla^2 f : \mathcal{S}_n \mapsto \mathrm{L}(\mathcal{S}_n, \mathcal{S}_n)$ such that

$$\lim_{\|H\|\to 0} \frac{\|\nabla f(X+H) - \nabla f(X) - \nabla^2 f(X)H\|}{\|H\|} = 0. \tag{A.1}$$

We show that $\nabla^2 f(X)H := -X^{-1}HX^{-1}$ satisfies (A.1). To this end, let $H \in \mathcal{S}_n$ be such that $(X+H)$ is invertible, and consider

$$\begin{aligned}
& \left\|\nabla f(X+H) - \nabla f(X) - \nabla^2 f(X)H\right\| \\
= \; & \left\|(X+H)^{-1} - X^{-1} + X^{-1}HX^{-1}\right\| \\
= \; & \left\|(X+H)^{-1}\left(I - (X+H)X^{-1} + (X+H)X^{-1}HX^{-1}\right)\right\| \\
= \; & \left\|(X+H)^{-1}\left(HX^{-1}HX^{-1}\right)\right\| \\
\leq \; & \left\|(X+H)^{-1}\right\| \|H\| \left\|X^{-1}HX^{-1}\right\|,
\end{aligned}$$

which shows that (A.1) indeed holds. □

A.2 Linear algebra lemmas

The following lemma is used in the proof of the sandwich theorem in Chapter 1.

Lemma A.2.1 *Let*

$$M_i := k\left(I_{\gamma_i} - J_{\gamma_i}\right), \quad i = 1, \ldots, k,$$

where I_{γ_i} is the $(\gamma_i \times \gamma_i)$ identity matrix, and J_{γ_i} the all-one matrix of the same size, for a given set of positive integers γ_i ($i = 1, \ldots, k$) with $\sum_{i=1}^{k} \gamma_i = n$.

Further define the block diagonal matrix

$$Y := \begin{pmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & M_k \end{pmatrix}. \tag{A.2}$$

One now has $\lambda_{\max}\left(Y + ee^T\right) = k$, where $e \in \mathbb{R}^n$ is the all-one vector.

Proof:

The Raleigh-Ritz theorem states that for any symmetric matrix A, one has:

$$\lambda_{\max}(A) = \max\left\{x^T A x \ : \ \|x\| = 1\right\}. \tag{A.3}$$

It follows that the maximal eigenvalue of Y is given by

$$\lambda_{\max}(Y) = \max\left\{\sum_{i=1}^{k} \alpha_i \lambda_{\max}(M_i),\ \sum_{i=1}^{k} \alpha_i = 1,\ \alpha_i \geq 0\ \forall i\right\}. \tag{A.4}$$

Moreover one has $\lambda_{\max}(M_i) = k$, so that (A.4) yields $\lambda_{\max}(Y) = k$. The eigenvector corresponding to k is orthogonal to the all-one vector e. To see this, note that $Yx = \lambda x$ implies

$$-k(\gamma_i - 1) \sum_{j \in C_i} x_j = \lambda \sum_{j \in C_i} x_j, \quad i = 1, \ldots, k,$$

so that $\sum_{j \in C_i} x_j = 0$ $(i = 1, \ldots, k)$ if $\lambda > 0$. In particular, $e^T x = 0$ from which it follows that k is also an eigenvalue of $Y + ee^T$. Assuming that k is not the largest eigenvalue of $Y + ee^T$, then the largest eigenvalue must have an eigenspace orthogonal to the eigenspace of k. The orthogonal complement of the eigenspace of k is spanned by the vectors

$$(x_{C_i})_j := \begin{cases} 1 & \text{if } j \in C_i \\ 0 & \text{otherwise,} \end{cases}$$

where $i = 1, \ldots, k$. The maximal eigenvalue of $Y + ee^T$ can therefore be computed from (A.3):

$$\begin{aligned}\lambda_{\max}\left(Y + ee^T\right) &= \max_x \left\{x^T\left(Y + ee^T\right)x \;:\; x \in \text{span}\left\{x_{C_1}, \ldots, x_{C_k}\right\},\ \|x\| = 1\right\} \\ &= \max_\alpha \left\{x^T Y x + \left(e^T x\right)^2 \;:\; x = \sum_{i=1}^k \alpha_i x_{C_i},\ \sum_{i=1}^k \gamma_i \alpha_i^2 = 1\right\}.\end{aligned}$$

Substituting the expression for x, and using the construction of Y simplifies this to

$$\begin{aligned}\lambda_{\max}\left(Y + ee^T\right) &= \max_\alpha \left\{-k\sum_{i=1}^k \alpha_i^2\left(\gamma_i^2 - \gamma_i\right) + \left(\sum_{i=1}^k \gamma_i\alpha_i\right)^2 \;:\; \sum_{i=1}^k \gamma_i\alpha_i^2 = 1\right\} \\ &= k + \max_\alpha \left\{-k\sum_{i=1}^k \left(\alpha_i\gamma_i\right)^2 + \left(\sum_{i=1}^k \gamma_i\alpha_i\right)^2 \;:\; \sum_{i=1}^k \gamma_i\alpha_i^2 = 1\right\}.\end{aligned}$$

The function to be maximized is always nonpositive, since it is of the form

$$-kz^Tz + (e^Tz)^2 \leq -kz^Tz + (\|e\|\|z\|)^2 = -kz^Tz + k\|z\|^2 = 0,$$

where $z_i = \alpha_i\gamma_i$, $(i = 1, \ldots, k)$. This leads to the contradiction $\lambda_{\max}\left(Y + ee^T\right) \leq k$.

We conclude that $\lambda_{\max}\left(Y + ee^T\right) = k$. □

The following lemma shows that the coefficient matrices arising in the solution of search directions in Chapters 3 to 6 are positive definite.

Lemma A.2.2 *Let $A_i \in \mathcal{S}_n$ $(i = 1, \ldots, m)$ be linearly independent, and let $Y, Z \succ 0$. The matrix $M \in \mathcal{S}_m$ with entries*

$$M_{ij} := \mathbf{Tr}\,\left(A_i Z A_j Y\right), \quad i, j = 1, \ldots, m$$

is positive definite.

Proof:

We prove that the quadratic form

$$q(x) = \sum_{i,j} M_{ij} x_i x_j$$

is strictly positive for all nonzero $x \in \mathbb{R}^n$. To this end, note that for given $x \neq 0$,

$$q(x) = \mathbf{Tr}\left(\left(\sum_i x_i A_i\right) Z \left(\sum_j x_j A_j\right) Y\right).$$

Denoting $A(x) := \sum_i x_i A_i$ (which is nonzero by the linear independence of the $A_i's$), one has:

$$q(x) = \mathbf{Tr}\ (A(x)ZA(x)Y) > 0,$$

where the inequality follows from $0 \neq A(x)ZA(x) \succeq 0$ and $Y \succ 0$. □

Next we give the proof of Lemma 3.3.3, which is used in the analysis of primal–dual affine scaling methods involving the 'condition number' centrality measure κ.

Lemma A.2.3 *Let $Q \succ 0$, and let $M \in \mathbb{R}^{n\times n}$ be skew–symmetric. One has $\det(Q+M) > 0$. Moreover, if it is known that $\lambda_i(Q+M) \in \mathbb{R}$ $(i = 1, \ldots, n)$, then*

$$0 < \lambda_{\min}(Q) \leq \lambda_{\min}(Q+M) \leq \lambda_{\max}(Q+M) \leq \lambda_{\max}(Q),$$

which implies $\kappa(Q+M) \leq \kappa(Q)$.

Proof:

First note that $Q+M$ is nonsingular since for all nonzero $x \in \mathbb{R}^n$:

$$x^T(Q+M)x = x^TQx > 0,$$

using the skew symmetry of M. We therefore know that

$$\psi(t) := \det[Q+tM] \neq 0 \quad \forall t \in \mathbb{R},$$

since tM remains skew–symmetric. One now has that ψ is a continuous function of t which is nowhere zero and strictly positive for $t = 0$ as $\det(Q) > 0$. This shows $\det(Q+M) > 0$.

To prove the second part of the lemma, assume $\lambda > 0$ is such that $\lambda > \lambda_{\max}(Q)$. It then follows that $Q - \lambda I \prec 0$. By the same argument as above we then have $(Q+M) - \lambda I$ nonsingular, or

$$\det\left((Q+M) - \lambda I\right) \neq 0.$$

This implies that λ cannot be an eigenvalue of $Q+M$. Similarly, $Q+M$ cannot have an eigenvalue smaller than $\lambda_{\min}(Q)$. This gives the required result. □

Appendix B

Notes on weighted centers

This is an appendix to Section 6.7.2, where concepts of weighted centers for SDP were discussed. The nonconvex weighted potential function discussed there is used here to analyse so-called target following methods for LP. It remains an open problem to extend this analysis to the more general SDP case.

B.1 Introduction

Medium and long step primal–dual interior–point methods in linear programming are of significant practical importance. Introduced by Kojima *et al.* [60], these methods have proven efficient in computational studies [71].

The worst-case complexity of long step algorithms with $O(1)$ barrier parameter (target) updates is $O(n \log 1/\epsilon)$ iterations, and for medium updates of $O(1/\sqrt{n})$ one has a worst–case bound of $O(\sqrt{n} \log 1/\epsilon)$ iterations [45, 39, 51]. Although the long step methods have a worse complexity bound than the short and medium step variants, the number of iterations performed in practice are often lower as becomes clear from the cited references.

Jansen *et al.* [46, 53, 52] provided a unifying framework of analysis for these important algorithms. Their 'target-following' approach involves choosing a series of targets to be approximated in the primal-dual space.

To introduce this approach, consider an LP problem in standard form

$$\min_x \left\{ c^T x \ : \ Ax = b, \ x \in \mathbb{R}^n_+ \right\},$$

where $\mathbb{R}^n_+$ denotes the positive orthant in $\mathbb{R}^n$, and its dual problem

$$\max_{y,s} \left\{ b^T y \ : \ A^T y + s = c, \ s \in \mathbb{R}^n_+, \ y \in \mathbb{R}^m \right\}.$$

For each target in the positive orthant, say $\bar{v} \in \mathbb{R}^n_+$, there exists a unique primal dual feasible pair (x, s) such that[1] $xs = \bar{v}^2$. Since all optimal pairs satisfy $xs = 0$, it is natural to choose a sequence of targets $\{\bar{v}^{(j)}\}$ in the positive orthant which converges to zero, and to compute a pair $(x^{(j)}, s^{(j)})$ such that $x^{(j)} s^{(j)} \approx (\bar{v}^{(j)})^2$ for each target in the sequence $\bar{v}^{(0)}, \bar{v}^{(1)}, \ldots$ Denoting $v^2 = xs$ for any primal-dual pair (x, s), we can make the approximation relation '$\approx$' more precise by using the proximity measure introduced by Jansen *et al.* in [53]:

$$\delta(v, \bar{v}) = \frac{1}{2 \min(\bar{v})} \left\| \frac{\bar{v}^2 - v^2}{v} \right\|,$$

where $\min(\bar{v}) := \min_{1 \le i \le n} \{\bar{v}_i\}$. We say v^2 is close to $\bar{v}^2$ if $\delta(v, \bar{v}) \le \tau$ for some tolerance $\tau < 1$.

The pair $(x^{(j)}, s^{(j)})$ is obtained by (approximately) solving the nonlinear system

$$\begin{aligned} Ax &= b, & x \ge 0 \\ A^T y + s &= c, & s \ge 0 \\ xs &= (\bar{v}^{(j)})^2. \end{aligned}$$

This is done iteratively by a damped Newton method, *i.e.* by taking damped Newton steps until the approximation condition is satisfied. The pairs $(x^{(j)}, s^{(j)})$ are called outer iterates and the points generated during the Newton process will be termed inner iterates.

The Newton step $(\Delta x, \Delta s)$ is obtained by solving the linearized system

$$\left.\begin{aligned} A\Delta x &= 0 \\ A^T \Delta y + \Delta s &= 0 \\ x\Delta s + s\Delta x &= (\bar{v}^{(j)})^2 - v^2, \end{aligned}\right\} \qquad \text{(B.1)}$$

[1] We use componentwise notation: xs indicates the vector obtained by multiplying the corresponding components of x and s, v^2 is the vector obtained by squaring the components of v, etc.

where the pair (x, s) is the last pair of inner iterates.

A damped Newton step $(\alpha\Delta x, \alpha\Delta s)$ with $\alpha \leq 1$ is used (as opposed to a full Newton step) and some care is required in choosing the step length α to ensure convergence of Newton's method.

To this end, a potential function is used in the analysis of the Newton process. The idea is that a sufficient reduction in the potential ensures proximity of the Newton iterates to the target $\bar{v}^{(j)}$. The analysis therefore reduces to analysing the effect of the damped Newton steps on the potential. (In practice the potential function may be used in line searches to do larger steps than allowed for by the analysis.) It is shown that a step length $\alpha \leq 1$ may be found at each step which ensures a decrease of the potential by an absolute constant.

The target $\bar{v}^{(j)}$ is updated as soon as the proximity condition is satisfied, *i.e.* as soon as the potential has been sufficiently decreased.

The result is the conceptually appealing *target following framework*:

Target following algorithm

Initialization
Given an initial feasible pair $(x^{(0)}, s^{(0)})$;
Let $\epsilon > 0$ be an accuracy parameter and $\tau < 1$ a proximity parameter;
Choose an initial target $\bar{v}^{(0)}$ such that $\delta(v^{(0)}, \bar{v}^{(0)}) \leq \tau$.
Set counter $j = 0$, $x = x^{(0)}$, and $s = s^{(0)}$.

While $(x^{(j)})^T s^{(j)} > \epsilon$ **do**

1. Solve the Newton equations (B.1) to obtain Δx and Δs.
2. Choose a suitable damping parameter (step length) $\alpha \leq 1$.
3. Set $x = x + \alpha\Delta x,\ s = s + \alpha\Delta s,\ v = \sqrt{xs}$;
4. If $\delta(v, \bar{v}^{(j)}) \leq \tau$ then
   - Let $(x^{(j+1)}, s^{(j+1)}) = (x, s)$;
   - Choose a new target $\bar{v}^{(j+1)}$;
   - Set $j = j + 1$.

Enddo.

The primal–dual potential function used in the papers [46, 53, 52] to determine the step length α is a strictly convex function. A new potential function is introduced here which is non-convex but still suitable for the complexity analysis of long step algorithms. The advantage of the new function is that it has an obvious analogy in the semidefinite programming case, whereas the potential used in [46, 53, 52] does not.

B.2 A new potential function

The new potential function used here is

$$f(x, s, \bar{v}) = \sum_{i=1}^{n} \left(x_i s_i \bar{v_i}^{-2} - 1 - \log x_i s_i \bar{v_i}^{-2} \right), \tag{B.2}$$

defined on the primal-dual feasible region. Using $v = \sqrt{xs}$ we can write (B.2) as

$$\phi(v, \bar{v}) = \sum_{i=1}^{n} \left(\frac{v_i^2}{\bar{v_i}^2} - 1 - \log \frac{v_i^2}{\bar{v_i}^2} \right). \tag{B.3}$$

Note that $\phi(v, \bar{v}) \geq \phi(\bar{v}, \bar{v}) = 0$.

The proposed potential function differs from the potential used by Jansen *et al.* in [53, 52, 46],

$$\tilde{f}(x, s, \bar{v}) = \frac{1}{\max(\bar{v}^2)} \left[\sum_{i=1}^{n} \left(x_i s_i - 1 - \bar{v_i}^{-2} \log x_i s_i \right) \right], \tag{B.4}$$

in that the 'weights' $\bar{v_i}$ are introduced in the duality gap term instead of the barrier term. The corresponding potential to (B.4) in terms of v is

$$\tilde{\phi}(v, \bar{v}) = \sum_{i=1}^{n} \frac{\bar{v_i}^2}{\max(\bar{v}^2)} \left[\frac{v_i^2}{\bar{v_i}^2} - 1 - \log \frac{v_i^2}{\bar{v_i}^2} \right]. \tag{B.5}$$

Notice that weighting factors $\frac{\bar{v}_i^2}{\max(\bar{v}^2)}$ appear in (B.5) which are absent from (B.3). Although the new formulation seems more natural it suffers from the apparent drawback that it is nonconvex, whereas $\tilde{f}$ in (B.4) is a strictly convex function of x and s for fixed $\bar{v}$.

Surprisingly, convexity is not a crucial issue here as the two potentials (B.2) and (B.4) have the same first order optimality conditions:

$$\left.\begin{array}{rcl} Ax & = & b \\ A^T y + s & = & c \\ xs & = & \bar{v}^2 \\ x, s & \geq & 0, \end{array}\right\} \tag{B.6}$$

which is simply the relaxed LP optimality conditions and known to have a unique solution (see *e.g.* [53]). Moreover it has already been indicated that the new potential attains its lower bound if $xs = \bar{v}^2$, proving existence of a unique minimizer of f. In other words, both potential functions have the solution of (B.6) as unique minimizer.[2]

A fixed $\bar{v}$ therefore represents a target which is approached by reducing the potential (B.2) using Newton's method. Once the potential has been sufficiently reduced, the target can be updated.

B.3 Reducing the potential

It remains to show that (B.2) can be successfully minimized by Newton's method. The next theorem shows that a damping parameter $\alpha \leq 1$ can always be found so that the damped Newton step reduces (B.2) by an absolute constant, determined by the current point (x, s) and the target $\bar{v}$ only.

Notation: The potential reduction will be given in terms of a function ρ of the distance $\delta(v, \bar{v})$ (where no confusion is possible we will use $\delta := \delta(v, \bar{v})$):

$$\rho(\delta) = \delta + \sqrt{1 + \delta^2}.$$

We will also borrow the following notation from Jansen [46]:

$$p_x \quad = \quad \frac{v \Delta x}{x}$$

[2] It is interesting to note that the new function does allow a convex reformulation in terms of variables $t_i = x_i s_i / \bar{v}_i^2$, and can be written as $\Psi(t) = \sum_{i=1}^n \Psi_i(t_i)$, with $\Psi_i(t_i) = t_i - 1 - \log t_i$.

$$\begin{aligned} p_s &= \frac{v\Delta s}{s} \\ p_v &= p_x + p_s = \frac{\bar{v}^2 - v^2}{v} \\ r &= \left\| \left[\frac{p_x}{v}, \frac{p_s}{v} \right] \right\|. \end{aligned}$$

Theorem B.3.1 *A damped Newton step* $(\alpha\Delta x, \alpha\Delta s)$ *with damping parameter*

$$\alpha = \frac{1}{r} - \frac{\max(\bar{v})^2}{\|p_v\|^2 + r\max(\bar{v}^2)} \leq 1 \tag{B.7}$$

gives a reduction of the potential function (B.2), bounded by

$$f(x, s, \bar{v}) - f(x + \alpha\Delta x, s + \alpha\Delta s, \bar{v}) \geq \frac{\delta^2\bar{\omega}^4}{\rho^2(\delta) + \rho(\delta)\delta\bar{\omega}^2}$$

where $\bar{\omega} = \frac{\max(\bar{v})}{\min(\bar{v})}$.

Proof:

By definition, the reduction of f is given by

$$\begin{aligned} \Delta f(\alpha) &\equiv f(x, s, \bar{v}) - f(x + \alpha\Delta x, s + \alpha\Delta s, \bar{v}) \\ &= -e^T \left(\alpha^2 \Delta x \Delta s \bar{v}^{-2} + \alpha x \Delta s \bar{v}^{-2} + \alpha \Delta x s \bar{v}^{-2} \right) + \sum_{i=1}^{n} \log\left(1 + \frac{\alpha\Delta x_i}{x_i}\right)\left(1 + \frac{\alpha\Delta s_i}{s_i}\right) \\ &= -e^T \left(\alpha^2 \frac{p_x p_s}{\bar{v}^2} + \alpha \frac{v}{\bar{v}^2}(p_x + p_s) \right) + \sum_{i=1}^{n} \log\left(1 + \frac{\alpha(p_x)_i}{v_i}\right)\left(1 + \frac{\alpha(p_s)_i}{v_i}\right) \\ &= -e^T \left(\frac{1}{2}\alpha^2 \left[\frac{p_v}{\bar{v}}\right]^2 - \frac{1}{2}\alpha^2 \frac{p_x^2 + p_s^2}{\bar{v}^2} + \alpha \frac{v}{\bar{v}^2} p_v \right) + \sum_{i=1}^{n} \log\left(1 + \frac{\alpha(p_x)_i}{v_i}\right)\left(1 + \frac{\alpha(p_s)_i}{v_i}\right) \end{aligned}$$

where e denotes the all–one vector and we have used $p_v = p_x + p_s$. The last term can be bounded by applying the inequality

$$\sum_{i=1}^{n} \log(1 + h_i) \geq \sum_{i=1}^{n} h_i + \|h\| + \log(1 - \|h\|) \quad \text{if } \|h\| < 1$$

to the combined vector $h = \left[\alpha\frac{p_x}{v}, \alpha\frac{p_s}{v}\right]$. Noting that $\|h\| = \alpha r$ in this case, we obtain

$$\Delta f(\alpha) \geq e^T \left(-\frac{1}{2}\alpha^2 \left[\frac{p_v}{\bar{v}}\right]^2 + \frac{1}{2}\alpha^2 \frac{p_x^2 + p_s^2}{\bar{v}^2} - \alpha \frac{v}{\bar{v}^2} p_v + \alpha \frac{p_v}{v} \right) + \alpha r + \log(1 - \alpha r)$$

$$\begin{aligned} &= e^T \left(-\frac{1}{2}\alpha^2 \left[\frac{p_v}{\bar{v}}\right]^2 + \frac{1}{2}\alpha^2 \frac{p_x^2 + p_s^2}{\bar{v}^2} + \alpha \left[\frac{p_v}{\bar{v}}\right]^2 \right) + \alpha r + \log(1-\alpha r) \\ &= e^T \left(\left(\alpha - \frac{1}{2}\alpha^2\right) \left[\frac{p_v}{\bar{v}}\right]^2 + \frac{1}{2}\alpha^2 \frac{p_x^2 + p_s^2}{\bar{v}^2} \right) + \alpha r + \log(1-\alpha r). \end{aligned}$$

The factor $\left(\alpha - \frac{1}{2}\alpha^2\right)$ is nonnegative, since $\alpha \leq 1$. We therefore have

$$\begin{aligned} \Delta f(\alpha) &\geq \frac{1}{\max(\bar{v}^2)} e^T \left(\left(\alpha - \frac{1}{2}\alpha^2\right) [p_v]^2 + \frac{1}{2}\alpha^2 (p_x^2 + p_s^2) \right) + \alpha r + \log(1-\alpha r) \\ &= \alpha \frac{\|p_v\|^2}{\max(\bar{v}^2)} + \alpha r + \log(1-\alpha r). \end{aligned}$$

The last expression is maximized by

$$\alpha^* = \frac{1}{r} - \frac{\max(\bar{v})^2}{\|p_v\|^2 + r\max(\bar{v}^2)} \tag{B.8}$$

which corresponds to

$$\Delta f(\alpha^*) \geq \frac{\|p_v\|^2}{r\max(\bar{v}^2)} - \log\left(1 + \frac{\|p_v\|^2}{r\max(\bar{v}^2)}\right). \tag{B.9}$$

The lower bound (B.9) on $\Delta f(\alpha^*)$ is obviously nonnegative but must be bounded away from zero. To accomplish this, note that expression (B.9) increases monotonically with $\frac{\|p_v\|^2}{r\max(\bar{v}^2)}$. We can therefore replace this quantity by a smaller value. Jansen [46] shows that

$$\frac{\|p_v\|^2}{r\max(\bar{v}^2)} \geq \frac{2\delta\bar{\omega}^2}{\rho(\delta)},$$

where $\bar{\omega} = \frac{\max(\bar{v})}{\min(\bar{v})}$. It follows that

$$\Delta f(\alpha^*) \geq \frac{\delta\bar{\omega}^2}{\rho(\delta)} - \log\left(1 + \frac{2\delta\bar{\omega}^2}{\rho(\delta)\delta}\right).$$

Using the inequality

$$x - \log(1-x) \geq \frac{x^2}{2(x+1)}$$

we arrive at the bound in the theorem statement. □

To fix our ideas, we choose a threshold value to decide when the current iterate v is 'close enough' to the current target $\bar{v}$. Following Jansen, we use $\tau = \frac{1}{4}$ as the threshold value. As long as $\delta(v, \bar{v}) > \frac{1}{4}$ we perform damped Newton steps with respect to $\bar{v}$ with the following guaranteed reduction of f each time:

Corollary B.3.1 *If* $\delta(v, \bar{v}) \geq \frac{1}{4}$ *then* $\Delta f \geq \frac{\bar{\omega}^4}{14+6\bar{\omega}^2}$.

The actual reduction obtained from a linesearch is of course much larger in general.

Once the proximity condition is satisfied, an upper bound on the potential is also known:

Lemma B.3.1 *If* $\delta(v, \bar{v}) \leq \frac{1}{4}$ *then* $\phi(v, \bar{v}) \leq \frac{2}{5}$.

Proof:

The potential ϕ in (B.3) can be written as

$$\begin{aligned} \phi(v, \bar{v}) &= \sum_{i=1}^{n} \left(\frac{v_i^2}{\bar{v}_i^2} - 1 \right) - \sum_{i=1}^{n} \log \frac{v_i^2}{\bar{v}_i^2} \\ &= \sum_{i=1}^{n} h_i - \sum_{i=1}^{n} \log(1 + h_i), \end{aligned}$$

where $h_i = \frac{v_i^2}{\bar{v}_i^2} - 1$. Since

$$\|h\| = \left\| \frac{v}{\bar{v}^2} \frac{e}{v} (v^2 - \bar{v}^2) \right\| \leq \left\| \frac{v}{\bar{v}^2} \right\|_\infty \|p_v\| \leq \rho(\delta) \left\| \bar{v}^{-1} \right\|_\infty 2 \min(\bar{v}) \delta = 2\delta\rho(\delta) < \frac{13}{20} < 1,$$

if $\delta \leq \frac{1}{4}$, one can use the inequality

$$\sum_{i=1}^{n} h_i - \sum_{i=1}^{n} \log(1 + h_i) \leq -\|h\| - \log(1 - \|h\|),$$

which holds if $\|h\| < 1$, to obtain

$$\phi(v, \bar{v}) \leq -2\delta\rho(\delta) - \log(1 - 2\delta\rho(\delta)).$$

Since $\delta(v, \bar{v}) \leq \frac{1}{4}$ and consequently $\delta\rho(\delta) < \frac{13}{40}$, we have $\phi(v, \bar{v}) \leq \frac{2}{5}$. □

All the tools necesserary to control the Newton process have now been developed, and we turn to the analysis of target updates.

B.4 Analysis of a general target update

Once the current iterate v is close enough to the target $\bar{v}$, *i.e.* $\delta(v,\bar{v}) \leq \frac{1}{4}$, the target can be updated to $\bar{v}^+$. The new potential $\phi(v,\bar{v}^+)$ can be bounded from above as follows:

Lemma B.4.1 *Given a current iterate v, current target $\bar{v}$, and target update $\bar{v}^+$, it holds that*

$$\phi(v,\bar{v}^+) \leq \phi(\bar{v},\bar{v}^+) + \max\left(\frac{\bar{v}^2}{(\bar{v}^+)^2}\right)\phi(v,\bar{v}) + 4\delta\rho(\delta)\sqrt{n}\max\left(\left|\frac{\bar{v}^2}{(\bar{v}^+)^2} - e\right|\right).$$

Proof:

The potential after the target update can be written as

$$\begin{aligned}
\phi(v,\bar{v}^+) &= \sum_{i=1}^{n}\left(\frac{v_i^2}{\bar{v}_i^2}\frac{\bar{v}_i^2}{(\bar{v}_i^+)^2}\right) - \sum_{i=1}^{n}\log\frac{\bar{v}_i^2}{(\bar{v}_i^+)^2} - \sum_{i=1}^{n}\log\frac{v_i^2}{\bar{v}_i^2} - n \\
&= \phi(\bar{v},\bar{v}^+) + \sum_{i=1}^{n}\left(\frac{v_i^2}{\bar{v}_i^2} - 1\right)\frac{\bar{v}_i^2}{(\bar{v}_i^+)^2} - \sum_{i=1}^{n}\log\frac{v_i^2}{\bar{v}_i^2} \\
&\leq \phi(\bar{v},\bar{v}^+) + \max\left(\frac{\bar{v}^2}{(\bar{v}^+)^2}\right)\phi(v,\bar{v}) + \sum_{i=1}^{n}\left(\frac{\bar{v}_i^2}{(\bar{v}_i^+)^2} - 1\right)\log\frac{v_i^2}{\bar{v}_i^2}.
\end{aligned}$$

In [46] (Theorem 4.3.6) it is proved that the last term is bounded by

$$\sum_{i=1}^{n}\left(\frac{\bar{v}_i^2}{(\bar{v}_i^+)^2} - 1\right)\log\frac{v_i^2}{\bar{v}_i^2} \leq 4\delta\rho(\delta)\sqrt{n}\max\left(\left|\frac{\bar{v}^2}{(\bar{v}^+)^2} - e\right|\right).$$

Substitution of this bound completes the proof. □

By combining Corollary B.3.1 and Lemma B.4.1 the following result is obtained.

Lemma B.4.2 *If the current iterate $v^{(j)}$ satisfies $\delta(v^{(j)},\bar{v}) \leq \frac{1}{4}$, and the target $\bar{v}$ is updated to $\bar{v}^+$, then at most*

$$\left\lceil \frac{14+6\bar{\omega}^2}{\bar{\omega}^4}\left[\phi(\bar{v},\bar{v}^+) + \frac{2}{5}\max\left(\frac{\bar{v}^2}{(\bar{v}^+)^2}\right) + \frac{13}{10}\sqrt{n}\max\left(\left|\frac{\bar{v}^2}{(\bar{v}^+)^2} - e\right|\right)\right]\right\rceil$$

damped Newton steps with respect to $\bar{v}^+$ are required to obtain an iterate $v^{(j+1)}$ satisfying $\delta(v^{(j+1)},\bar{v}^+) \leq \frac{1}{4}$.

B.5 Complexity analysis for Dikin–type target updates

All that remains is to choose a target updating scheme. Consider for example the Dikin-type updates introduced by Jansen *et al.* [49, 53, 52]:

$$\bar{v}^+ = \bar{v}\left(e - \frac{\theta}{\max(\bar{v}^{2\nu})}\bar{v}^{2\nu}\right) \tag{B.10}$$

with $0 < \theta < 1/(2\nu+1)$. Note that $\nu = 0$ corresponds to weighted path following methods. Furthermore an initial choice $\bar{v}^{(0)} = \mu e$ for some fixed $\mu > 0$ leads to a central path following algorithm.

We can bound the number of Newton steps necessary to approximate a new target $\bar{v}^+$ given $\delta(v, \bar{v}) \leq \frac{1}{4}$ by providing bounds for each of the terms in Lemma B.4.2.

Lemma B.5.1 *Let $\bar{v}^+$ be a new target obtained by updating the old target $\bar{v}$ via (B.10). We then have the bounds*

$$\phi(\bar{v}, \bar{v}^+) \leq \frac{3n\theta^2}{(1-\theta)^2}, \quad \max\left(\frac{\bar{v}^2}{(\bar{v}^+)^2}\right) \leq \frac{1}{(1-\theta)^2}, \quad \max\left(\left|\frac{\bar{v}^2}{(\bar{v}^+)^2} - e\right|\right) \leq \frac{\theta(2-\theta)}{(1-\theta)^2}.$$

Proof:

The last two inequalities follow from the observation

$$\frac{\bar{v}_i^2}{(\bar{v}_i^+)^2} = \frac{1}{\left(1 - \theta\bar{v}_i^{2\nu}/\max\left(\bar{v}^{2\nu}\right)\right)^2} \leq \frac{1}{(1-\theta)^2}.$$

The bound on $\phi(\bar{v}, \bar{v}^+)$ is obtained as follows

$$\begin{aligned}
\phi(\bar{v}, \bar{v}^+) &\equiv \sum_{i=1}^n \frac{\bar{v}_i^2}{(\bar{v}_i{}^+)^2} - \sum_{i=1}^n \left(\log \frac{\bar{v}_i^2}{(\bar{v}_i{}^+)^2} - 1\right) \\
&= \sum_{i=1}^n \frac{1}{\left(1 - \theta\bar{v}_i^{2\nu}/\max\left(\bar{v}^{2\nu}\right)\right)^2} - \sum_{i=1}^n \left(\log \frac{1}{\left(1 - \frac{\theta\bar{v}_i^{2\nu}}{\max(\bar{v}^{2\nu})}\right)^2} - 1\right) \\
&= \sum_{i=1}^n \frac{1}{\left(1 - \frac{\theta\bar{v}_i^{2\nu}}{\max(\bar{v}^{2\nu})}\right)^2} \left[1 + 2\left(1 - \frac{\theta\bar{v}_i^{2\nu}}{\max\left(\bar{v}^{2\nu}\right)}\right)^2 \left(\log\left(1 - \frac{\theta\bar{v}_i^{2\nu}}{\max\left(\bar{v}^{2\nu}\right)}\right)\right.\right.
\end{aligned}$$

$$- \left(1 - \frac{\theta \bar{v}_i^{2\nu}}{\max\left(\bar{v}^{2\nu}\right)}\right)^2\Bigg)\Bigg].$$

Using $\log(1-x) \leq -x$ if $x < 1$ and simplifying, we have

$$\begin{aligned}
\phi(\bar{v}, \bar{v}^+) &\leq \sum_{i=1}^{n} \frac{1}{\left(1 - \frac{\theta \bar{v}_i^{2\nu}}{\max(\bar{v}^{2\nu})}\right)^2} \left[2\left(1 - \frac{\theta \bar{v}_i^{2\nu}}{\max\left(\bar{v}^{2\nu}\right)}\right)^2 \left(\frac{-\theta \bar{v}_i^{2\nu}}{\max\left(\bar{v}^{2\nu}\right)}\right)^2 \right. \\
&\quad \left. + 2\frac{\theta \bar{v}_i^{2\nu}}{\max\left(\bar{v}^{2\nu}\right)} - \frac{\theta^2 \bar{v}_i^{4\nu}}{\max\left(\bar{v}^{4\nu}\right)}\right] \\
&\leq \frac{1}{(1-\theta)^2} \sum_{i=1}^{n} \left(3\frac{\theta^2 \bar{v}_i^{4\nu}}{\max\left(\bar{v}^{4\nu}\right)} - 2\frac{\theta^3 \bar{v}_i^{6\nu}}{\max\left(\bar{v}^{6\nu}\right)}\right) \leq \frac{3n\theta^2}{(1-\theta)^2}.
\end{aligned}$$

□

We now have a bound on how many damped Newton steps are required to reach the proximity of a new target:

Corollary B.5.1 *Assume that $\delta(v, \bar{v}) \leq \frac{1}{4}$. If the target is updated to $\bar{v}^+$ using the target updating scheme (B.10), then at most*

$$O\left(\frac{1}{(\bar{\omega}^+)^4}\left(n\theta^2 + \sqrt{n}\theta\right)\right)$$

damped Newton steps are needed to approximate $\bar{v}^+$, where $\bar{\omega}^+ = \frac{\max(\bar{v}^+)}{\min(\bar{v}^+)}$.

The last question is how many target updates are required to obtain an ϵ–approximate solution. It is simple to prove the following (see [46]):

Lemma B.5.2 *Let a primal-dual starting pair $(x^{(0)}, s^{(0)})$ be given. Choose the first target as $\bar{v}^{(0)} = v^{(0)}$. After at most*

$$O\left(\frac{1}{\theta \omega_0^{2\nu}} \log \frac{(x^{(0)})^T s^{(0)}}{\epsilon}\right)$$

target updates using (B.10) the algorithm terminates with a primal dual pair (x^*, s^*) such that $(x^*)^T s^* \leq \epsilon$.

Combining these results, we obtain the complexity bound for the complete algorithm:

Theorem B.5.1 *The target following algorithm requires at most*

$$O\left(\frac{n\theta+\sqrt{n}}{\omega_0^{2\nu+4}}\log\frac{(x^{(0)})^T s^{(0)}}{\epsilon}\right)$$

damped Newton steps for convergence.

A large target update with $\theta = O(1)$ therefore requires fewer than $O(n/\omega_0^{2\nu+4})$ Newton steps, whereas medium step methods with $\theta = O(1/\sqrt{n})$ require fewer than $O(\sqrt{n}/\omega_0^{2\nu+4})$ steps.

These complexity bounds are the same as those obtained by using the standard convex potential function. We conclude that the nonconvex potential (B.3) is a proper alternative to the usual convex logarithmic barrier potential.

B.6 Further work

It has already been mentioned that the new potential function (B.2) has an extension to the SDP case.

Recall that the SDP problem (P) in standard form is defined as

$$\begin{aligned}\min\ & \mathbf{Tr}\,(CX)\\ \mathbf{Tr}\,(A_iX) &= b_i \quad i=1,\ldots,m\\ X &\succeq 0\end{aligned}$$

where the A_i's and C are symmetric matrices, and that the optimality conditions for (P) are

$$\begin{aligned}\mathbf{Tr}\,(A_iX) &= b_i \qquad i=1,\ldots,m\\ \textstyle\sum_{i=1}^m y_iA_i + S &= C\\ XS &= 0\\ X, S &\succeq 0.\end{aligned}$$

Note that the potential function

$$f(X, S, \bar{V}) = \mathbf{Tr}\,(XS\bar{V}^{-2}) - \log\det\left(\frac{1}{2}(XS + XS)\bar{V}^{-2}\right) - n$$

is a natural extension of the new LP potential (B.2) to the semi-definite case, where $\bar{V}$ is a symmetric positive definite 'target matrix'. To the best of our knowledge this is the first weighted potential function for semidefinite programming. Extension of the analysis in the previous sections would therefore broaden the target following framework to semi-definite programming. This is the subject of further research.

Appendix C

Search directions for the embedding problem

Conditions for the existence and uniqueness of several search directions for the self–dual embedding problem of Chapter 2 (Section 2.4) are derived here.

Recall that the search directions for the embedding problem (2.21) (Section 2.4) are defined by

$$\left.\begin{array}{lllllll} & \mathbf{Tr}\,(A_i\Delta X) & -\Delta\tau b_i & +\Delta\theta\bar{b}_i & & & = 0 \quad \forall i \\ -\sum_{i=1}^m \Delta y_i A_i & & +\Delta\tau C & -\Delta\theta\bar{C} & -\Delta S & & = 0 \\ b^T\Delta y & -\mathbf{Tr}\,(C\Delta X) & & +\Delta\theta\alpha & & -\Delta\rho & = 0 \\ -\bar{b}^T\Delta y & +\mathbf{Tr}\,(\bar{C}\Delta X) & -\Delta\tau\alpha & & & -\Delta\nu & = 0 \end{array}\right\} \tag{C.1}$$

and

$$\left.\begin{array}{rcl} H_P\,(\Delta X S + X\Delta S) & = & \mu I - H_P(XS), \\ \rho\Delta\tau + \tau\Delta\rho & = & 2\mu - \tau\rho \\ \nu\Delta\theta + \theta\Delta\nu & = & 2\mu - \theta\nu, \end{array}\right\} \tag{C.2}$$

where

$$\bar{b}_i \;:=\; b_i - \mathbf{Tr}\,(A_i), \quad i = 1, \ldots, m$$

$$\begin{aligned}\bar{C} &:= C - I \\ \alpha &:= 1 + \mathbf{Tr}\,(C),\end{aligned}$$

and H_P is the linear transformation given by

$$H_P(M) := \frac{1}{2}\left[PMP^{-1} + P^{-T}M^TP^T\right],$$

for any symmetric matrix M, and where the *scaling matrix* P determines the symmetrization strategy. The choices of P which were listed in Table 1.1 in Chapter 1 are reproduced here in Table C.1 for the sake of convenience. We prove (or

P	Reference
$\left[X^{\frac{1}{2}}\left(X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}}$	Nesterov and Todd [85];
$X^{-\frac{1}{2}}$	Monteiro [73], Kojima *et al.* [65];
$S^{\frac{1}{2}}$	Monteiro [73], Helmberg *et al.* [44], Kojima *et al.* [65];
I	Alizadeh, Haeberley and Overton [4];

Table C.1: Choices for the scaling matrix P.

derive sufficient conditions) for existence and uniqueness of the search directions corresponding each of the choices of P in Table C.1. To this end, we will write the equations (C.1) and (C.2) as a single linear system and show that the coefficient matrix of this system is nonsingular. The approach used here follows the analysis by Todd *et al.* in [108], where this result was proved for SDP problems in the standard form (P) and (D). The same result can also be obtained from the general analysis by Shida *et al.* [101]. The class of problems considered in [101] actually includes the self-dual embedding described in this thesis. However, by extending the analysis of Todd *et al.* we also show how the search direction for the embedding can be computed, and avoid the mathematically involved analysis in [101].

We introduce the notation

$$\tilde{X} := \begin{bmatrix} X & & \\ & \tau & \\ & & \theta \end{bmatrix}, \quad \tilde{S} := \begin{bmatrix} S & & \\ & \rho & \\ & & \nu \end{bmatrix}, \quad \tilde{A}_i := \begin{bmatrix} A_i & & \\ & -b_i & \\ & & \bar{b}_i \end{bmatrix} \ (i = 1, \ldots, m),$$

and define $\tilde{P}$ by replacing X by $\tilde{X}$ and S by $\tilde{S}$ in Table C.1. We will rewrite (C.2) in terms of the following *symmetric Kronecker product notation*:

- $\mathbf{svec}(X) := [X_{11},\ \sqrt{2}X_{12},\ \ldots\ \sqrt{2}X_{1n},\ X_{22},\ \sqrt{2}X_{23},\ \ldots,\ X_{nn}]^T$;
- The *symmetric Kronecker product* $G \otimes_s H$ of $G, H \in \mathbb{R}^{n\times n}$ is implicitly defined via

$$(G \otimes_s K)\,\mathbf{svec}(H) := \frac{1}{2}\mathbf{svec}\left(KHG^T + GHK^T\right)\ (\forall\ H \in \mathcal{S}_n).$$

Using the symmetric Kronecker notation, we can combine (C.1) and (C.2) as

$$\begin{bmatrix} 0 & \tilde{\mathcal{A}} & 0 \\ -\tilde{\mathcal{A}}^T & \mathcal{S}_{kew} & I \\ 0 & E & F \end{bmatrix} \begin{bmatrix} \Delta y \\ \mathbf{svec}\left(\Delta\tilde{X}\right) \\ \mathbf{svec}\left(\Delta\tilde{S}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{svec}\left(\mu I - H_{\tilde{P}}(\tilde{X}\tilde{S})\right) \end{bmatrix} \tag{C.3}$$

where

$$\begin{aligned} \tilde{\mathcal{A}} &:= [\mathbf{svec}(\tilde{A}_1)\ \ldots\ \mathbf{svec}(\tilde{A}_m)]^T \\ \mathcal{S}_{kew} &:= \begin{bmatrix} 0 & \mathbf{svec}(C) & -\mathbf{svec}(\bar{C}) \\ -\mathbf{svec}(C)^T & 0 & \alpha \\ \mathbf{svec}(\bar{C})^T & -\alpha & 0 \end{bmatrix} \\ E &:= \tilde{P} \otimes_s (\tilde{P}^{-T}\tilde{S}),\quad F := (\tilde{P}\tilde{X}) \otimes_s \tilde{P}^{-T}. \end{aligned}$$

The following lemma provides more information about the matrices E and F.

Lemma C.0.1 (Todd *et al.* [108]) *Let $\tilde{P}$ be invertible and $\tilde{X}$ and $\tilde{S}$ symmetric positive definite. Then the matrices E and F are invertible. If one also has $H_{\tilde{P}}(\tilde{X}\tilde{S}) \succ 0$ then the symmetric part of $E^{-1}F$ is also positive definite.*

We are now in a position to prove a sufficient condition for uniqueness of the search direction.

Theorem C.0.1 *The linear system (C.3) has a unique solution if $H_{\tilde{P}}(\tilde{X}\tilde{S}) \succ 0$.*

Proof:

We consider the homogeneous system

$$\begin{bmatrix} 0 & \tilde{\mathcal{A}} & 0 \\ -\tilde{\mathcal{A}}^T & \mathcal{S}_{kew} & I \\ 0 & E & F \end{bmatrix} \begin{bmatrix} \Delta y \\ \mathbf{svec}\left(\Delta\tilde{X}\right) \\ \mathbf{svec}\left(\Delta\tilde{S}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{C.4}$$

and prove that it only has the zero vector as solution.

From (C.4) we have

$$\mathbf{svec}\left(\Delta\tilde{S}\right) = \tilde{\mathcal{A}}^T\Delta y - \mathcal{S}_{kew}\mathbf{svec}\left(\Delta\tilde{X}\right)$$

and

$$\mathbf{svec}\left(\Delta\tilde{S}\right) = -F^{-1}E\mathbf{svec}\left(\Delta\tilde{X}\right). \tag{C.5}$$

Eliminating $\mathbf{svec}\left(\Delta\tilde{S}\right)$ from the last two equations gives

$$\tilde{\mathcal{A}}^T\Delta y - \mathcal{S}_{kew}\mathbf{svec}\left(\Delta\tilde{X}\right) + F^{-1}E\mathbf{svec}\left(\Delta\tilde{X}\right) = 0. \tag{C.6}$$

System (C.4) also implies

$$\tilde{\mathcal{A}}\mathbf{svec}\left(\Delta\tilde{X}\right) = 0. \tag{C.7}$$

From (C.6) we have

$$\begin{aligned} \mathbf{svec}\left(\Delta\tilde{X}\right)^T\tilde{\mathcal{A}}^T\Delta y - \mathbf{svec}\left(\Delta\tilde{X}\right)^T\mathcal{S}_{kew}\mathbf{svec}\left(\Delta\tilde{X}\right) & \\ +\mathbf{svec}\left(\Delta\tilde{X}\right)^T F^{-1}E\mathbf{svec}\left(\Delta\tilde{X}\right) &= 0. \end{aligned}$$

The first term on the left hand side is zero, by (C.7), and the second term is zero by the skew-symmetry of $\mathcal{S}_{kew}$. We therefore have

$$\mathbf{svec}\left(\Delta\tilde{X}\right)^T F^{-1}E\mathbf{svec}\left(\Delta\tilde{X}\right) = 0,$$

which shows that $\Delta\tilde{X} = 0$, since EF^{-1} is assumed to be (non-symmetric) positive definite. It follows that $\Delta\tilde{S} = 0$ by (C.5). Furthermore, $\Delta y = 0$ by (C.6), since $\tilde{\mathcal{A}}$ has full rank (the matrices A_i $(i = 1, \ldots, m)$ are linearly independent). □

All that remains is to analyse the condition

$$H_{\tilde{P}}(\tilde{X}\tilde{S}) \succ 0 \tag{C.8}$$

in the theorem. For the first three choices of $\tilde{P}$ in Table C.1, condition (C.8) always holds. For $\tilde{P} = I$ (the so-called AHO direction), (C.8) becomes the condition $\tilde{X}\tilde{S} + \tilde{S}\tilde{X} \succ 0$. It is shown in [76] and [64] that this latter condition holds in the following neighbourhood of the central path:

$$\left\{(\tilde{X}, \tilde{S}) \ : \ \left\| \frac{1}{2\mu}(\tilde{X}\tilde{S} + \tilde{S}\tilde{X}) - I \right\| < 1, \ \tilde{X} \succ 0, \ \tilde{S} \succ 0 \right\},$$

for some $\mu > 0$.

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Samenvatting

Inwendige Punt Methoden voor Semidefiniet Programmeren
door
Etienne de Klerk

Het onderzoek naar inwendige punt methoden voor optimalisatieproblemen begon in 1979 met resultaten van Khachian over de (polynomiale) complexiteit van lineaire programmering (LP). In 1984 ontwierp Karmarkar een algoritme met verbeterde complexiteit, en beweerde dat dit sneller werkte dan de traditionele simplex methode. Sindsdien wordt er actief onderzoek gedaan naar inwendige punt methoden.

In het begin van de jaren negentig werd de klasse van problemen, die oplosbaar zijn met deze methoden, uitgebreid met sommige convexe niet-lineaire optimalisatieproblemen. Onder die problemen vallen onder meer semidefiniete programmering problemen (SDP); dit zijn problemen met lineaire doelfuncties en lineaire nevenvoorwaarden zoals bij LP, maar met positief semidefiniete matrices als variabelen (in plaats van niet-negatieve vectoren bij LP). SDP problemen zijn al lang bekend vanwege vele toepassingen in o.a. regeltechniek, grafentheorie, en globale optimalisatie.

In Hoofdstuk 2 wordt een techniek voor de initialisatie van SDP algoritmen beschreven, namelijk de zelf–duale herformulering van het probleem. Hier doen zich een aantal problemen voor, omdat de dualiteitstheorie van SDP zwakker is dan die van LP. Er wordt echter aangetoond hoe men – ondanks deze complicaties – vast kan stellen of een gegeven probleem toelaatbaar is en hoe men eventueel een optimale waarde kan bepalen. Hier is de sterkere *extended Lagrange Slater* dualiteitstheorie (ELSD) van Ramana voor nodig, i.p.v. de meer gebruikelijke Lagrange dualiteitstheorie.

Hoofdstuk 3 generaliseert twee primaal–duaal inwendige punt methoden van LP naar SDP. Het gaat hier om zogenaamde *affiene schalingsmethoden*. Deze methoden staan sterk in de belangstelling sinds het duidelijk werd dat Karmarkar's algoritme verwant was aan de primale affiene schalingsmethode van Dikin uit 1967. De *primaal–duaal affiene*

schalingsmethode van Monteiro e.a. en de *primaal–duaal Dikin affiene schalingsmethode* van Jansen e.a. worden in Hoofdstuk 3 uitgebreid naar SDP, en dezelfde iteratie complexiteit wordt bewezen als in het LP-geval. Er zijn verschillende manieren om de zoekrichtingen van de algoritmen te veralgemeniseren van LP naar SDP; bij het uitbreiden wordt hier gebruikt gemaakt van de symmetrische schaling van Nesterov en Todd. Dit is cruciaal, aangezien sommige andere uitbreidingen niet globaal convergent zijn.

De methoden van Hoofdstuk 3 zijn uit theoretisch oogpunt belangrijk, maar zijn in deze vorm niet geschikt voor implementatie. In Hoofdstuk 4 wordt er daarom onderzoek gedaan naar implementeerbare varianten van die algoritmen. De resulterende methoden vallen binnen het kader van primaal–duaal potentiaal reductie methoden en hebben een 'predictor–corrector' aard. De nieuwe methoden concurreren goed met de potentiaal reductie methoden van Nesterov en Todd, zoals geïmplementeerd door Vandenberghe en Boyd; dit blijkt uit een vergelijkende studie op drie testsets van problemen.

De laatste twee hoofdstukken zijn gewijd aan zogenaamde padvolgende methoden, die ook bekend staan als logaritmische barriere methoden. Deze methoden dateren eigenlijk uit de jaren '60 in het werk van Fiacco en McCormick, en werden 'herontdekt' naar aanleiding van Karmarkar's algoritme. In Hoofdstuk 5 wordt de primale methode behandeld. Een eenvoudige analyse van de globale lineaire convergentie wordt gepresenteerd. Verder wordt aangetoond hoe de methode gebruikt kan worden in samenhang met de zelf–duale herformulering van Hoofdstuk 2. De primaal–duale methoden komen aan de orde in Hoofdstuk 6. Een verfijnde analyse van deze methoden wordt gepresenteerd, en de analyse maakt het mogelijk om de bekende methoden van Jiang en van Nesterov en Todd op een eenvoudige manier te analyseren. Verder worden manieren besproken om de convergentie van primaal–duale padvolgende methoden op te krikken d.m.v. adaptieve strategieën.

Dit proefschrift wordt afgesloten met een overzicht van verkregen resultaten en suggesties voor nader onderzoek in Hoofdstuk 7.

Curriculum Vitae

Etienne de Klerk was born on December 24th, 1968 in Pretoria, South Africa. He finished high school at the *Hoërskool Waterkloof* in Pretoria in 1986. In 1989 he completed a BSc (cum laude) degree at the University of Pretoria with as main subjects applied and pure mathematics and physics. He was awarded the Pierre du Plessis prize for best achievements in Physics III. During his undergraduate studies he also worked part-time at the Atomic Energy Corporation of South Africa. In 1990 he completed the degree BSc(Hons) (cum laude) in Applied Mathematics at the abovementioned university, and worked as student-assistant in the Department of Mathematics. The period 1991–1992 was devoted to a MSc(Mechanics) degree in the Department of Mechanical Engineering of the University of Pretoria, with a masters thesis entitled *New Parallel Solvers for Discretized Heat and Fluid Flow Equations*. In 1993 he worked as a research assistant in the Structural Optimization Research Group (SORG) of the Department of Mechanical Engineering.

The year 1994 saw the author of this thesis relocate to Delft in order to pursue PhD studies on interior point algorithms in the Department of Statistics, Stochastics, and Operations Research of the Delft University of Technology. These studies spanned the period January 1994 to December 1997. He was also involved in a joint project with the Interfacultary Reactor Institute on optimal nuclear reactor reloading strategies in the same period.